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RESEARCH MEMORANDUM

AN EXERCISE IN WELFARE ECONOMICS (III)

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Abstract

The present paper summarizes first the headlines of setting up our preferred alternative determination model for preference structures i.e. that of the theoretical DSID-model that was already highlighted in earlier papers. However, the main purpose of the present paper is to discuss the inverse technique in a more profound way and to show 'how and why' we introduced the Moore-Penrose Inverse technique into the final framework of the theoretical DSID- as well as of the DSID-application model.

Section I. Introduction

Three earlier papers and especially the second research memorandum in the series under the title of 'An exercise in Welfare Economics' reviewed on and evaluated the many efforts of establishing collective preferences.[1] Much attention was devoted to a broad discussion of the a priori as well as the ex post approaches. The a priori concept clearly appears to be second to the ex post one. The latter starts from the implicit preferences idea and recognizes that planning behavior involves constrained optimization conditional on expectations of the future while constraints and the information set are relatively well understood. Moreover, it takes into account policy decisions do not follow simple repeated optimalizations as the framework of the reaction function approach suggests. Two alternative determination models for implicit preference structures were highlighted in that same paper.[2] We argued that the (D)eterministic (S)tatic (I)mplicit (D)etermination model takes care of results not being influenced anymore by the a priori functional form of the preference function. The problems of the second order conditions for an optimum could have been overcome too by the use of the concept of the relative preference elasticity. The second alternative of the '(I)nteractive (R)especification' model focused on the reconstruction, by a pseudo-simulation of a given policy choice, of the characteristics of planning behavior and the underlying preference structure in more detail compared with the D.S.I.D.-model approach. Nevertheless, the aforementioned second research memorandum concluded that the I.R.-way of doing may deliver reasons perhaps why the D.S.I.D.-approach has to be preferred, especially if the attention is merely turned to test stability through time of retrospective relative preference elasticities rather than to simulate the planning behavior in itself.

The present third research memorandum in the same series summarizes first and for shortness sake, the headlines of setting up our preferred alternative determination model for preference structures i.e. that of the theoretical D.S.I.D.-model that was already highlighted in the foregoing research memoranda. However, the main purpose of the present paper is to discuss the inverse technique in a more profound way.

Section II. Setting up a new determination model for preference structures

Acceptance of the ex post approach for empirical investigations on the relevant preference structures about which policy management has been carried out by the responsible policy makers in the past and in different countries, plays the most prominent part in the establishment of the theoretical D.S.I.D.-model in earlier papers and the present one as well as for setting up the D.S.I.D.-application model in the research memorandum 'An exercise in welfare economics (IV)'. The ultimate availability of these so-called "Deterministic-Static-Implicit-Determination"-models should deliver more insight into stability of the retrospective preference elasticities of the relevant target- and instrumental variables of the economy concerned (with respect to each other) during a number of years of the observation horizon. May be a measure of constancy of the elasticity of relative preferences of the relevant variables with respect to the most important one could be computed in order to integrate this in programming and optimal control exercises in future research.
Polynomial curve-fitting of the computed preference elasticity ratio data can give more insight in the dynamic properties of the relative preference structures and into the possibility of the mentioned integration to above.

This section will devoted to a short review of the theoretical D.S.I.D.-model as a start to the derivation of the D.S.I.D.-application model. We shall revisit the theoretical and practical implications of the underlying idea of the ex post measure of the "social" preference structure of a national economy. Some important problems boil down to special assumptions to be made in the ex post approach as regards the supposed optimal quantitative economic policy problem.

Three main elements corresponding with these special assumptions can be distinguished:

1. Assumptions about the specifications of the preference function $\omega(\underline{y},\underline{z},t)$ and of the side-conditions

$$f_n(\underline{y},\underline{z},\underline{v},\underline{x},\underline{y}_{-},\underline{v}_{-},\underline{e},t) = 0;\ n = 1,\dots,N.$$

2. Assumptions about the optimizing problem as a whole and its inverse.

3. The mathematical programming technique assumed to be performed for computation of the solution of the optimizing problem and of its inverse.

II.A. Assumptions about the preference function, the structure of the side-conditions, the mathematical programming procedure as concerns the optimizing problem of quantitative economic policy and its inverse

In the research memorandum preceding to the present one we saw that methodological reasons, theoretical as well as practical ones, have had great influence on our main objective in this complete exercise to solve problems c) and d) in preference to a simultaneous solution of all four original questions a), b), c) and d), stated there.
The effort to attain this objective together with requirements of consistency and the existence of the interdependence between the assumptions to be made as regards the aforementioned distinction of the main elements referred to above, determine which exact assumptions about the preference function, the side-conditions and the mathematical programming technique have been made finally and are subject-matter of the discussions in this section. The same can be said about the assumptions to be made relating to the establishment of the D.S.I.D.-application model. Also than we are focusing on the solution of questions c) and d) listed in the mentioned research memorandum. But because of the existence of the quivalence property as concerns the solutions of the two different concepts for the D.S.I.D.-model caused by different assumptions in the theoretical-respectively in the application model about the specifications of the preference function, makes it easier in the latter model to compute the numerical values of the relative preference elasticity ratios i.c. relative, parameter ratios of ex post computed numerical values of parameters of an a priori specified preference function (that of the Cobb-Douglastype), knowing the econometric specification of the side-conditions and the observed results of an assumed optimal quantitative economic policy.

Hereafter we shall revisit first the assumptions of the theoretical D.S.I.D.-model.

§ II.A.1. Specification of the preference functions and the structure of the side-conditions

Setting of the theoretical D.S.I.D.-model we only postulated a scalar-valued preference function:

$$\omega(\underline{y},\underline{z},t) \tag{II.A.1.a}$$

which is assumed to be continuously differentiable in the neighboorhood of the constrained maximum to be found in the original optimal quantitative economic policy problem.

The argument vector $(\underline{y},\underline{z})$ consists of two subvectors $\underline{y}$ and $\underline{z}$ being the (J×1)-target vector respectively the (K×1)-instrumental vector of economic policy in a certain year t (t = 1,...,T). So the argument vector $(\underline{y},\underline{z})$ is of order J + K, defined in an J + K dimensional euclidean space. Its first J elements are $y_{j,t}$ (j = 1,...,J) and its last K elements are $z_{k,t}$ (k = 1,...,K), $\forall_t^{1,\ldots,T}$ denoting the target variables and the instrument variables respectively. Nothing more is assumed as regards the specification of the preference function but we think of it as a function for which first order optimum conditions are necessary and sufficient in order to speak about a constrained global maximum of this function.

We proved that under certain general circumstances such a function $\omega(\underline{y},\underline{z},t)$ always exists.[3)]

In this case the a priori specification is only confined to assumptions about target- and instrumental i.e. time dependent variables containing in the preference function and its availability to be globally maximized within a certain range, only imposing on it first order optimum conditions. The question which variables should be entered in the preference function depends finally on the contents of the specified economic model composing the set of constraints of the optimizing problem.

For convenience's sake we shall assume that a national economy can be described by a set of one period lagged linear equations. At it will be

clear enough further on this circumstance does not constrain the generality of the theoretical D.S.I.D.-model conclusions in cases where economies should be described in other ways as long as they rendered by equalities. Also if one is dealing with non-linearity and one of the possible alternatives of dynamization of economic models the theoretical determination model is remaining relevant.
The economic model that we shall use in next paragraph is the linear-dynamic model stated already in the preceding research memorandum for $p = 1$. However in the present case the coefficient matrices and the original constant term vector are thought to be time dependent. So we are dealing with lagged endogenous variables of one year and get for $f_n(\underline{y},\underline{z},\underline{v},\underline{x},\underline{y}_-,\underline{v}_-,\underline{e},t) = 0;\ n = 1,\ldots,N$

$$A_t\underline{y}_t + B_t\underline{v}_t + C_t\underline{z}_t + D_t\underline{x}_t + A_{1,t}\underline{y}_{t-1} + B_{1,t}\underline{v}_{t-1} + \underline{e}_t = \underline{0} \qquad \text{(II.A.1.b)}$$

where A_t, B_t, C_t, D_t $A_{1,t}$ and $B_{1,t}$ are known coefficient matrices; $\underline{y}_t$, $\underline{v}_t$, $\underline{z}_t$, $\underline{x}_t$, $\underline{y}_{t-1}$ and $\underline{v}_{t-1}$ are the different vectors of variables of the model. $\underline{e}_t$ and $\underline{0}$ are the known dynamical original constant term- respectively the nullvectors of the model ($t = 1,\ldots,T$).
Corresponding orders and meanings of the other matrices and vectors are already stated. Here we are dealing with already known feasible elements of the argument vector $(\underline{y}.\underline{z})$ in function (II.A.1.a). They are explicitely stated in the economic model (II.A.1.b). This explicit knowledge, which answers the question what the feasible target- and instrumental variables are to be contained in the preference function is from a practical point of view only possible if we are dealing with the a priori optimation problem of the a priori approach discussed in the preceding research memorandum. Theoretically, exact knowledge of the elements to be integrated in the preference function is possible. For this we need to be sure about which targets and instruments the policy decision-unit took its decisions in the past. But that was just one of the points why we accepted the ex post approach in order to derive the preference structures of the policy decision-unit viz. because of the impossibility to have honest and/or correct information about the various policy objectives and instruments.

In the a priori approach where the possibility of correct and honest information about these objectives and instruments is one of the essential assumptions, one tries to get this information by interviewing the responsible policy decision-unit or scanning of public documents. Together with the knowledge of the relevant economic model one is able to see which of the endogenous economic respectively controllable exogenous economic variables of the economic model can be used for a justified translation of the really, honestly and correctly exposed targets and instruments into terms of the available target- and instrumental variables of the economic model.
Setting up again the theoretical D.S.I.D.-model and accepting the underlying ex post approach we use the same a priori optimization idea relating to the existence of certain target- and instrumental variables about which one has decided in the past.
However exact knowledge about which of the exogenous variables and of the available instrumental variables of the economic model of which the econometric specification is supposed to be known, could be considered as the right translation variables for the real targets and instruments of optimal economic policy used in the past is not available and should be subject of investigation.
For convenience' sake and considering the establishment of the theoretical D.S.I.D.-model as a good demonstration of the line of thought underlying the D.S.I.D.-application model respectively the advantages and shortcomings of the latter we maintain the economic model specification of system (II.A.1.b) and assume for this case the theoretical possibility of being sure about which targets and instruments the policy decision-unit took its decisions in the past.
We have to realize however this is only allowed in theoretical exercises. This circumstance determines some of the questions of the inverted optimization problem in the sense we defined already in preceding papers. They will be one of the subject-matters of the next paragraph.

§ II.A.2. The optimizing problem of quantitative economic policy and its inverse

Before being able to accept the Theil-approach where it is assumed that optimal economic policy to be performed in a national economy could right away be exposed by means of the optimal quantitative economic policy idea, many considerations and assumptions were to be made. We discussed them broadly in aforementioned papers.[4)] There, questions were discussed relating to the theoretical and practical possibility to perform optimal economic policy in a country.

An affirmative answer to these questions demands for the existence of de facto performance of optimal economic policy and in which way one could realize this. Besides the main determinants of such optimal decision processes and their dynamic properties were subject-matter of discussion.

Finally we accepted the idea of optimal quantitative economic policy and its resulting optimizing problem as a good starting point in our ex post approach for reasons already stated herefore.

The optimizing scheme relates de facto to quantitative economic planning by a national (or international) policy decision-unit based on ranked preference-orderings on possible values of explicitely expressed target- and instrumental variables mathematically formalized by a preference function.

The possibilities are delineated by means of an economic model. In this case the preference function is thought of as the objective function to be maximized subject to side-conditions of the economic model. If we take into account the assumption we made about the specification of the preference function and the structure of the side-conditions in § II.A.1 we get for the optimizing scheme:

$$\forall_t^{t=1,\dots,T}$$

$$\max_{(\underline{y},\underline{z},t)\in V(t)} \omega(\underline{y},\underline{z},t) \qquad \text{(II.A.2.a)}$$

$$V(t) = \{(\underline{y},\underline{z},t)/(\underline{y},\underline{z},t) \in E^{J+K},\ A_t\underline{y}_t + B_t\underline{v}_t + C_t\underline{z}_t + D_t\underline{x}_t +$$

$$+ A_{1,t}\underline{y}_{t-1} + B_{1,t}\underline{v}_{t-1} + \underline{e}_t = \underline{0}\}$$

where $\omega(\underline{y},\underline{z},t)$ and $V(t)$ have the same properties as stated in foregoing paragraph.

The de facto a priori optimization problem (II.A.2.a) indicates that before performing the maximization procedure for a certain year t by means of an available mathematical programming technique the values of the non-controlled exogenous variables $x_{1,t}$ $(1 = 1,\ldots,L)$, being elements of the $(L\times1)$ - $\underline{x}_t$ vector, the values of the lagged endogenous variables $\underline{y}_{j,t-1}$ $(j = 1,\ldots,J)$ and $\underline{v}_{i,t-1}$ $(i = 1,\ldots,I)$, being elements of the $(J\times1)$ - $\underline{y}_{t-1}$ resp. the $(I\times1)$ - $\underline{v}_{t-1}$ vectors and the values of the dynamic constant term vector $\underline{e}_t$ are known. Therefore we speak in this case of an explicit, static determination model for calculating optimal values of $\underline{y}_t$, $\underline{z}_t$ and $\underline{v}_t$ in year t in the sense of maximizing preference function $\omega(\underline{y},\underline{z},t)$.

In the ex post approach, disregarding the problems mentioned relating to the possible knowledge about which targets and instruments the policy decision-unit took its decisions in the past, we start with the same scheme (II.A.2.a). Whereas in the de facto a priori optimization problem the assumptions of § II.A.1 only compose part of all the assumptions to be made (f.i. in this case it is also assumed that the mathematical resp. econometric shape of the preference function is known) in the ex post approach. In addition to the assumptions of § II.A.1 the optimal values of $\underline{y}_t$, $\underline{z}_t$ and $\underline{v}_t$ are assumed to be known for every year t of the observation horizon.

Of course the same is assumed here as regards the vectors $\underline{x}_t$, $\underline{y}_{t-1}$, $\underline{v}_{t-1}$, $\underline{e}_t$ and $\underline{0}$.

Now the de facto a priori optimization problem is transformed into the inverted optimization problem i.e. the inverse of the scheme (II.A.2.a) is relevant.

If we denote the observed results being conceived of as optimal by the index (0), we can write instead of (II.A.2.a) the analogous scheme of <u>the inverted optimization problem</u> in the following interrogative way: which are the numerical values of the relevant characteristics of $\omega(\underline{y},\underline{z},t)$ satisfying the conditions:

$$\forall_t^{t=1,\ldots,T}$$

$$\underset{(\underline{y}(0),\underline{z}(0),t)\in V(t)}{\mathrm{Max}} \omega(\underline{y},\underline{z},t) = \omega(\underline{y}(0),\underline{z}(0),t) \qquad \text{(II.A.2.b)}$$

$$V(t) = \{(\underline{y}(0),\underline{z}(0),t)/(\underline{y}(0),\underline{z}(0),t) \in E^{J+K},\ A_t\underline{y}(0)_t + B_t\underline{v}(0)_t +$$

$$+ C_t\underline{z}(0)_t + D_t\underline{x}(0)_t + A_{1,t}\underline{y}(0)_{t-1} + B_{1,t}\underline{v}(0)_{t-1} + \underline{e}_t = \underline{0}\}$$

where $\omega(\underline{y},\underline{z},t)$ and $V(t)$ have the same properties as stated in (II.A.2.a).

Taking into account the performed mathematical programming technique depending on the nature of mathematical tractability of system (II.A.2.b), we are dealing with the implicit, static determination model for calculating empirical values of the relevant characteristics of the preference function for every year t of the observation horizon T ($t = 1,\dots,T$).
These results are the data on which investigations on dynamic properties of the relevant characteristics can be based.

Deriving scheme (II.A.2.b) we used the arbitrary certainty-assumption about which target- and instrumental variables were policy objectives of the policy decision-unit in the past.
However one of the reasons to accept the ex post approach was to find out in a less arbitrary way which ones of the possible target- and instrumental variables were de facto relevant objectives for policy-making in the past. Within the limitation of the assumed econometric model (II.A.1.b), all the endogenous variables being elements of the two vectors $\underline{v}_t$ and $\underline{y}_t$ could be conceived of as real target-variables respectively as irrelevant variables in a certain year t. The analogous possibility is true relating to the set of possible instrumental variables being elements of the vector $\underline{z}_t$. Therefore in the inverted case focusing on generality as much as possible, <u>all</u> the a priori possible target- and instrumental variables, the number of them being determined by the assumed econometric model, should be integrated in the argument vector of the preference function of scheme (II.A.2.b); so we get the following interrogative scheme:

Which are the numerical values of the relevant characteristics of $\omega(\underline{y},\underline{v},\underline{z},t)$ satisfying the conditions:

$$\forall_t^{t=1,\dots,T}$$

$$\max_{(\underline{y}(0),\underline{v}(0),\underline{z}(0),t)\in V(t)} \omega(\underline{y}.\underline{v},\underline{z},t) = \omega(\underline{y}(0),\underline{v}(0),\underline{z}(0),t) \quad \text{(II.A.2.c)}$$

$$V(t) = \{(\underline{y}(0),\underline{v}(0),\underline{z}(0),t)/(\underline{y}(0),\underline{v}(0),\underline{z}(0),t) \in E^{N+K},\ A_t\underline{y}(0)_t + B_t\underline{v}(0)_t + C_t\underline{z}(0)_t + D_t\underline{x}(0)_t + A_{1,t}\underline{y}(0)_{t-1} + B_{1,t}\underline{v}(0)_{t-1} + \underline{e}_t = \underline{0}\}$$

where $\omega(\underline{y},\underline{v},\underline{z},t)$ and $V(t)$ have the same properties as stated in (II.A.2.a). Besides in scheme (II.A.2.c) it is supposed that the vector of non-controlled exogenous variables $\underline{x}_t$ does not contain elements belonging to the set of possible instruments for a certain year t of the observation horizon. With the strive for generality as denoted by scheme (II.A.2.c) it often happens that a solution of this inverted optimization problem is possible in a theoretical-consistent sense, whereas in practice this solution cannot or can hardly be found.

One of the reasons for this problem concerns the availability of a suitable mathematical programming technique, respectively of its corresponding arithmetical procedure for solution computation of the inverted optimization problem. It is possible too that we can dispose of suitable mathematical programming - and arithmetical computational techniques to get rid of the practical solution problem, whereas the capacity of the available computer (software) fails in order to perform the desired computation in a satisfactory way. Such circumstances ask for alternatives. They boil down to accept other assumptions as regards the preference function, the side-model and/or the whole optimization problem itself. Development of other econometric models and/or as yet unknown mathematical programming respectively arithmetical computational techniques can get us to a mathematically and theoretical-economically tractable situation where a consistent combination of preference function, side-model, mathematical programming- and arithmetical computational technique is available. The latter way of doing sometimes implies truncation of the contents and/or of the form of the preference function.

Indication about the influence of truncation can be got by performing sensitivity-analysis.

Switching of alternatives as regards truncation and considering the corresponding solution results can give insight into mutual dependence and

relevance of the target- and instrumental variables not only for the solution results of scheme (II.A.2.c) but also for the assumed underlying actual economic policy-management in the past.
From this it is clear that sensitivity-analysis is always justified also in those cases where computer (software) capacity does not fail.

§ II.A.3. The mathematical programming procedure for solution of the optimizing problem and its inverse

Evaluating the line of thought of the ex post approach in foregoing paragraph we mentioned the various theoretical and practical problems to be solved concerning mathematical tractability and theoretical economic-political consistency of the system (II.A.2.c). All the necessary adjustments in order to get a mathematical and theoretical-economical tractable situation must be feasible within the framework of the mathematically translated procedure of the original economic policy problem discussed at the beginning of the foregoing paragraph. Considering again scheme (II.A.2.c) a suitable mathematical programming technique to be assumed could be the Lagrangean optimization technique. Theoretically this possibility can cause difficulties viz. in cases where the observed values of the elements of the target and instrumental vectors represent optimal corner solutions. In such circumstances the Lagrange optimization procedure must be substituted by the Kuhn-Tucker-optimization procedure. How we get rid of this difficulty will be stipulated elsewhere.[5)] There will be shown that for our purposes of empirical investigation the Lagrange technique is the suitable one to be assumed because optimal corner solutions did not appear in the investigated situations; the constraints of the econometric models were continuously differentiable in the neighbourhood of the observed maxima and besides those were all stated as equations with equality sign. So they all correspond with the scheme's of foregoing paragraph. At this moment we have all information needed to derive again the theoretical "Deterministic-Static-Implicit-Determination"-model in next section.

II.B. The theoretical "Deterministic-Static-Implicit-Determination"-model (D.S.I.D.-model) for Preference structures

§ II.B.1. Introduction

As we stated in section II.A the establishment of the theoretical D.S.I.D.-model in this section should be considered as a good demonstration of the line of thought underlying the D.S.I.D.-application model and of the advantages and shortcomings of the latter.
Using it for these objectives it will be allowed to assume in this theoretical case the possibility of being sure about which targets and instruments a policy decision unit took their decisions in the past. This will be done in spite of the consequences such a postulation has for the formulation of the inverted optimization problem and its solution as we saw herefore.

So practically we are dealing with the schemes (II.A.2.a) and (II.A.2.b) connecting with the optimum problem of quantitative economic policy and its inverse.
Because meanings and orders of the different matrices and vectors were given there, we don't need to restate them if we use them again in next paragraphs. Only meanings and orders will be stated in so far we use matrices and vectors otherwise than we did herefore.

§ II.B.2. A synthesis of the Lagrange Multiplier and the Generalized Inverse techniques

A good start will be the postulated optimum problem of quantitative economic policy denoted by scheme (II.A.2.a).
As already said under certain circumstances the Lagrange multiplier technique can be the suitable optimization procedure to be performed for seeking of the desired solution of problem (II.A.2.a).
We establish the following Lagrange function:

$$\forall_t^{t=1,\dots,T}$$

$$L_t = \omega_t(\underline{y}_t,\underline{z}_t) - \lambda_t'\{A_t\underline{y}_t + B_t\underline{v}_t + C_t\underline{z}_t + D\underline{x}_t + A_{1,t}\underline{y}_{t-1} +$$

$$+ B_{1,t}\underline{v}_{t-1} + \underline{e}_t\} \qquad \text{(II.B.2.a)}$$

where L_t denotes the Lagrange function for every year t of the considered horizon of T years.

Taking the first partial derivatives of this function L_t with respect to the vector elements corresponding to the target-, instrumental and irrelevant variables of economic policy respectively and the Langrangean multipliers setting them equal to zero results in the first-order Lagrange optimization relations:

$$\forall_t^{t=1,\dots,T}$$

$$\frac{\partial L_t}{\partial \underline{y}_t} = \underline{\omega}^*_{\underline{y}_t}(\underline{y}_t,\underline{z}_t) - A_t'\underline{\lambda}_t = \underline{o}_3$$

$$\frac{\partial L_t}{\partial \underline{z}_t} = \underline{\omega}^*_{\underline{z}_t}(\underline{y}_t,\underline{z}_t) - C_t'\underline{\lambda}_t = \underline{o}_4 \qquad \text{(II.B.2.b)}$$

$$\frac{\partial L_t}{\partial \underline{v}_t} = \underline{o}_1 \qquad - B_t'\underline{\lambda}_t = \underline{o}_5$$

$$\frac{\partial L_t}{\partial \underline{\lambda}_t} = \underline{o}_2 \qquad - A_t\cdot\underline{y}_t - B_t\cdot\underline{v}_t - C_t\cdot\underline{z}_t - D_t\underline{x}_t - A_{1,t}\underline{y}_{t-1} - B_{1,t}\underline{v}_{t-1} - \underline{e}_t = \underline{o}_6$$

where A_t', C_t' and B_t' are the transposed original A_t, C_t and B_t-matrices of the Lagrange function (II.B.2.a).

$\underline{o}_3$ is an $(J\times 1)$-nullvector, $\underline{o}_4$ is an $(K\times 1)$-nullvector and $\underline{o}_5$ is an $(I\times 1)$-nullvector. The other vectors and matrices are already known. From the Jacobian matrix to be derived from the constraints of scheme (II.A.2.a) it will be clear that the conditions for a global maximum are fulfilled. In this case the first-order Lagrange conditions in (II.B.2.b) guarantee that we deal with the maximization problem of scheme (II.A.2.a).

The same is true as regards its inverse, formulated in scheme (II.A.2.b). In the latter case ex post knowledge of the realized values of the targets, irrelevant, instrumental, non-controllable and of the lagged endogenous policy variables, satisfying the original a priori numerically-specified economic model relations, provides the situation where the corresponding specified economic model relations in (II.B.2.b) boil down to null-relations, and will allow us to single them out from this system. Doing so it will produce the linear and homogeneous system of equations corresponding to the inverted optimization problem in next subparagraph. As will be clear in the other next subparagraphs, solutions of the latter system are found in nearly all cases by means of a special concept of the Generalized Inverse technique. Taking into account the demonstrated use of the Lagrange Multiplier technique, these two techniques are the main elements of our establishment of the ultimate D.S.I.D.-model. So titling this paragraph as "A synthesis of the Lagrange Multiplier and the Generalized Inverse techniques" has been justified.

§ II.B.2.1. <u>The linear and homogeneous system of equations of the inverted optimization problem</u>

In the same way as we started with scheme (II.A.2.a) for denoting the optimizing problem of quantitative economic policy postulated by us, we can use now scheme (II.A.2.b) as a good starting point for further reasoning. As already said, the linear and homogeneous system of equations of the inverted optimization problem can be derived from system (II.B.2.b) taking into account the observed realized results of the different policy variables as indicated by the index (o) in scheme (II.A.2.b). Singling out the last N-null-equations of system (II.B.2.b) we derive:

$$\forall_t^{t=1,\dots,T}$$

$$\begin{aligned}
&\underline{\omega}^*_{\underline{y}_t}(\underline{y}(o)_t,\underline{z}(o)_t) - A_t'\underline{\lambda}_t = \underline{o}_3 \\
&\underline{\omega}^*_{\underline{z}_t}(\underline{y}(o)_t,\underline{z}(o)_t) - C_t'\underline{\lambda}_t = \underline{o}_4 \qquad\qquad (\text{II.B.2.1.a}) \\
&\underline{o}_1 \qquad\qquad\qquad - B_t'\underline{\lambda}_t = \underline{o}_5 .
\end{aligned}$$

System (II.B.2.1.a) contains the unknown vectors $\underline{\omega}^{*}_{\underline{y}_t}(\underline{y}(o)_t,\underline{z}(o)_t)$, $\underline{\omega}^{*}_{\underline{z}_t}(\underline{y}(o)_t,\underline{z}(o)_t)$ and $\underline{\lambda}_t$; the first two consisting of the elements which denote the as-yet unknown realized marginal preferences of the different target- and instrumental variables, whereas the latter is the Lagrange Multiplier-vector. Explicit distinction of these unknown elements with the known ones will be realized in its corresponding matrix-presentation in the next subparagraph.

§ II.B.2.2. Matrix presentation of the inverted optimization problem

Rearranging and writing system (II.B.2.1.a) in matrix-notation we get the following matrix-presentation of the inverted optimization problem:

$$\forall_t^{t=1,\dots,T}$$

$$\begin{bmatrix} \mathcal{I} & -A'_t \\ & -C'_t \\ \mathcal{O} & -B'_t \end{bmatrix} \cdot \begin{bmatrix} \underline{\omega}^{*}_{\underline{y}_t}(\underline{y}(o)_t,\underline{z}(o)_t) \\ \underline{\omega}^{*}_{\underline{z}_t}(\underline{y}(o)_t,\underline{z}(o)_t) \\ \underline{\lambda}_t \end{bmatrix} = \begin{bmatrix} \underline{o} \end{bmatrix} \qquad \text{(II.B.2.2.a)}$$

where $\mathcal{I}$ is the $(J+K) \times (J+K)$-Identitymatrix; $\mathcal{O}$ is an $(I) \times (J+K)$ null-matrix.

In system (II.B.2.2.a) we have partitioned all the elements of the original linear and homogeneous system (II.B.2.1.a) into two groups viz. the group of known elements represented by the left-side matrix, to be denoted hereafter with $\mathcal{A}_t$, and the group of unknown elements represented by the left-side vector which we shall denote with $\underline{p}_t$. So instead of (II.B.2.2.a) we can write:

$$\forall_t^{t=1,\dots,T}$$

$$\mathcal{A}_t \cdot \underline{p}_t = \underline{o} \qquad \text{(II.B.2.2.b)}$$

where $\mathcal{A}_t$ is an (N+K) × (N+J+K)-matrix, $\underline{p}_t$ is an (N+J+K) × 1-vector and $\underline{o}$ is an (N+K) × 1-null-vector.

All the derivations in this section performed herefore relate to the postulated optimization problem of quantitative economic policy respectively to its inverse formulated in the schemes (II.A.2.a) and (II.A.2.b).
Because they relate also every time to one single period of time (a year), system (II.B.2.2.b) is always underdetermined in the unknowns of vector $\underline{p}_t$ if the preference function $\omega_t(\underline{y},\underline{z},t)$ contains at least one target variable i.e. J > 0.
Taking into account this last difficulty together with one of our main objectives, viz. the determination of the realized relative preferences and the realized relative preference elasticities as regards all target- and instrumental variables, we normalize system (II.B.2.2.b) in order to be able to find meaningful results by means of the Moore-Penrose inverse technique to be discussed in subparagraph II.B.2.6.
This normalization can be performed by singling out the last element from the vector $\underline{p}_t$, setting equal to the arbitrary base-value of one. Its multiplication with the last column-elements of the matrix $\mathcal{A}_t$ and transferring the resulting vector to the right side of the system and changing it of sign delivers an (N+K) × 1 vector $\underline{q}_t$. If we denote the curtailed matrix $\mathcal{A}_t$ and vector $\underline{p}_t$ with $\bar{\mathcal{A}}_t$ and $\bar{\underline{p}}_t$, we derive:

$$\forall_t^{t=1,\dots,T}$$

$$\bar{\mathcal{A}}_t \cdot \bar{\underline{p}}_t = \underline{q}_t \qquad \text{(II.B.2.2.c)}$$

$$\forall_t^{t=1,\dots,T}$$

$$\begin{bmatrix} \mathcal{I} - \bar{A}_t' \\ - \bar{C}_t' \\ \mathcal{O} - \bar{B}_t' \end{bmatrix} \cdot \begin{bmatrix} \underline{\omega}^*_{\underline{y}_t}(\underline{y}(o)_t,\underline{z}(o)_t) \\ \underline{\omega}^*_{\underline{z}_t}(\underline{y}(o)_t,\underline{z}(o)_t) \\ \bar{\underline{\lambda}}_t \end{bmatrix} = \begin{bmatrix} \underline{a}'_{\ell,t} \\ \underline{c}'_{\ell,t} \\ \underline{b}'_{\ell,t} \end{bmatrix}$$

where $\underline{q}_t$ consists of the three subvectors, $\underline{a}'_{\ell,t}$, $\underline{c}'_{\ell,t}$ and $\underline{b}'_{\ell,t}$, being the last columnvectors of the original matrices A'_t, C'_t and B'_t of system (II.B.2.2.a).
The curtailed concepts of the latter are denoted with $\bar{A}'_t$, $\bar{C}'_t$ and $\bar{B}'_t$. $\bar{\underline{\lambda}}_t$ is the curtailed vector $\underline{\lambda}_t$ of Lagrange-multipliers, where its last element has been put equal to one i.e. $\lambda_{N,t} = 1$, $\forall_t^{t=1,\dots,T}$.
From (II.B.2.2.b) and (II.B.2.2.c) we can say that the latter system will also be underdetermined if the number of target-variables of the preference function exceeds one i.e. $J > 1$.

§ II.B.2.3. Consistency, inconsistency and the corresponding solutions of the inverted optimization problem

As we saw in section II.A and § II.B.2 we disregard stochastic disturbance terms in the a priori specified economic model, the side-conditions of the postulated optimization model and its inverse.
Besides the conditions of a global maximum are fulfilled, and the ex post known realized values of the target, irrelevant, instrumental, uncontrollable and of the lagged endogenous policy variables, satisfy the a priori specified economic model. Taking into account this situation we can infer from it the consistency of the system (II.B.2.b)/(II.B.2.2.c) i.e. the consistency of the inverted optimization model and of the corresponding normalized system (II.B.2.2.a) expressed by (II.B.2.2.c).
In nearly all cases only the theory of the "Generalized Inverse" will enable us to get meaningful, economic relevant solutions of the inverted optimization problem, in so far as they can give us ultimately good basis information in order to detect the dynamic properties over time of the preference structure of a certain policy decision-unit which is our principal objective in this study. As regards this last objective, our purpose will be selection of a vector of ratio-values, being a vector solution of systems (II.B.2.2.a) and (II.B.2.2.b) as well as of system (II.B.2.2.c). Doing the same thing in the D.S.I.D.-application model, we can derive immediately the 'realized' relative preference elasticities of the target- and instrumental variables with respect to each other for every $t = 1,\dots,T$, and through this their stability through time during the observation horizon of T years. However we must take care that solutions to be

found are to be very sensitive to variations in the known elements referring to a certain year t which biases them against the H_o-hypothesis: "Stability of the preference structure". Actually it means that a solution to system (II.B.2.2.c) should deliver a numerically valued $\bar{p}_t$-vector which is very sensitive for the numerically known elements of the matrix $\mathcal{A}_t$ and the vector g_t, $\forall_t^{t=1,\ldots,T}$. As will be shown hereafter the Moore-Penrose concept of the Generalized Inverse-technique can take care of this claim. Getting ahead of matters we are dealing with furtheron, we can say that our problem will be to find out how many vectors $\bar{p}_t$ of system (II.B.2.2.c) are, $\forall_t^{t=1,\ldots,T}$, simultaneously satisfying this system and the aforementioned condition of sensitivity. In the aforeconcluded case of consistency there exist at least one vector $\bar{p}_t$ satisfying this system of equations, otherwise the system would be inconsistent. Although we shall discuss in the next subparagraphs the general notion of the "Generalized Inverse", and the main distinct concepts of it, we shall denote now the resulting "solutions" of (II.B.2.2.c) using the Moore-Penrose definition in the two cases to be distinguished i.e.

Case I. <u>Consistency of system (II.B.2.2.c)</u>

$\forall_t^{t=1,\ldots,T}$

$g_t \in R(\mathcal{A}_t) \leftrightarrow \mathcal{A}_t.\bar{p}_t = g_t$ has a solution which can be formulated as:

$$\bar{p}_{t,o} = \mathcal{A}_t^{+} g_t + (I - \mathcal{A}_t^{+}\mathcal{A}_t)\underline{r} \qquad \text{(II.B.2.3.a)}$$

where $\bar{p}_{t,o}$ is a "general" (N+J+K-1) × (1)-Least-Squares-Solution (L.S.S.)- and optimum-vector.

$\mathcal{A}_t^{+}$ is the (N+J+K-1) × (N+K) Moore-Penrose inverted matrix $\mathcal{A}_t$ and $\underline{r}$ is an arbitrarily real-valued (N+J+K-1) × (1) vector. $\bar{p}_{t,o}$ is called a "general" L.S.S.- and optimum-vector because:

$$\forall_{\underline{r}}^{\underline{r}\in E^{N+J+K-1}} : \min_{\bar{p}_t} |\mathcal{A}_t \bar{p}_t - g_t|^2 = |\mathcal{A}_t \bar{p}_{t,o} - g_t|^2 = 0 \qquad \text{(II.B.2.3.b)}$$

These mathematical expressions explain the theorem that the inverted optimization problem, respectively its corresponding mathematical system (II.B.2.2.c), has a solution if and only if the vector g_t is an element of the "Range" of the matrix $\mathcal{A}_t$.
If we are dealing with J > 1 target variables in the preference function the rank of matrix $\mathcal{A}_t$ will be lower than (N+J+K-1). This means that system (II.B.2.2.c) has an infinite number of solutions denoted by (II.B.2.3.a). However if the preference function contains J = 1 target variable, $\mathcal{A}_t$ is an (N+K) × (N+K)-matrix. Here a matrix rank of N + K means non-singularity and the general solution procedure of (II.B.2.3.a) boils down to the normal inverse procedure. In this case the Moore-Penrose inverse $\mathcal{A}_t^+$ is identical with the Normal inverse $\mathcal{A}_t^{-1}$ and the null-space of matrix $\mathcal{A}_t$ denoted by $(I-\mathcal{A}_t^+ . \mathcal{A}_t)\underline{r}$ in (II.B.2.3.a) contains only the (N+K) × (1) null-vector. As we shall see hereafter the presence of an infinite number of solutions $\bar{p}_{t,o}$ will be relevant in our D.S.I.D-model. Therefore stating case II of inconsistency of system (II.B.2.2.c) must be regarded as for the sake of theoretical completeness i.e.

Case II. <u>Inconsistency of system (II.B.2.2.c)</u>

$\forall_t^{t=1,\dots,T}$

$g_t \not\in R(\mathcal{A}_t)$ and $\mathcal{A}_t \, \bar{p}_t = g_t$ has no solutions.

In this case the vector g_t is not an element of the "Range" of the matrix $\mathcal{A}_t$ and so system (II.B.2.2.c) has no solutions. However it can be proved that the best approximation of the optimal solution vector $\bar{p}_{t,o}$ is found by using the Moore-Penrose Inverse of the matrix $\mathcal{A}_t$ viz.

$$\bar{p}_{t,\hat{o}} = \mathcal{A}_t^+ . g_t \qquad \text{(II.B.2.3.c)}$$

$\bar{p}_{t,\hat{o}}$ is called the "Best Approximate" Solution-vector (B.A.S.-vector) because it is the vector having the least "Norm" for which we get the least error

$$\underline{e}_t = \mathcal{A}_t \cdot \bar{p}_t - g_t.$$

We can find it by minimizing the sum of squares of the deviations between $\mathcal{A}_t \cdot \bar{p}_t$ and g_t. So we get:

$\forall_t^{t=1,\dots,T}$

$$\min_{e_{i,t}} \sum_1 e_{i,t}^2 = \min_{\underline{e}_t} \underline{e}_t' \cdot \underline{e}_t = \min_{\bar{p}_t} |\mathcal{A}_t \cdot \bar{p}_t - g_t|^2 =$$

$$\min_{\bar{p}_t} (\mathcal{A}_t \cdot \bar{p}_t - g_t)' \cdot (\mathcal{A}_t \cdot \bar{p}_t - g_t) =$$

$$(\mathcal{A}_t \, \bar{p}_{t,\hat{o}} - g_t)'(\mathcal{A}_t \, \bar{p}_{t,\hat{o}} - g_t)$$

We declared already case I: "Consistency of system (II.B.2.2.c)" together with an infinite number of solutions $\bar{p}_{t,o}$ will be actual for our D.S.I.D.-model.

In II.B.2.7 and §'s II.B.3 and II.B.4 we shall show that the aforementioned sensitivity-conditions to be satisfied by $\bar{p}_{t,o}$-vector with respect to variations of the $\mathcal{A}_t$ and g_t-elements, can be found using the Moore-Penrose inverse-properties. It boils down to a selection of that unique solution vector $\bar{p}_{t,u}$ from the set of feasible solution vectors $\bar{p}_{t,o}$ formulated in (II.B.2.3.a) which satisfies the minimum Euclidean norm. The minimum Euclidean norm can be defined now as:

$$\min_{\underline{r}} |\bar{p}_{t,o}|_2 \qquad \text{(II.B.2.3.d)}$$

It can be proved that aforementioned unique solution vector $\bar{\underline{p}}_{t,u}$ is found as follows:

$$\bar{\underline{p}}_{t,u} = \min_{\underline{r}} \|\bar{\underline{p}}_{t,o}\|_2 = \mathcal{A}_t^{+} g_t \qquad \text{(II.B.2.3.e)}$$

Considering (II.B.2.3.b) and (II.B.2.3.e) it will be clear that we call $\bar{\underline{p}}_{t,u}$ a Least-Least-Square-Solution (L.L.S.S.)-vector. Choice of the minimum Euclidean Norm can be justified taking into account the condition of sensitivity. Comparison with other mostly used selection procedures learns: that the L.L.S.S.-procedure is a most useful one. This will be demonstrated in II.B.4.

Finishing this sub-paragraph we remark that in case I as well as in case II the "unique" solution vector is found in the same way i.e. by means of the multiplication of the Moore-Penrose Inverse of matrix $\mathcal{A}_t$ with the g_t-vector of system (II.B.2.2.c).

§ II.B.3. The Generalized Inverse

The synthesis of the Lagrange Multiplier- and the Generalized Inverse-technique achieved in the D.S.I.D.-model means the basic theory underlying this model consists of main elements of the basic theory underlying mathematical programming as well as of the basic theory underlying the concept of the generalized inverse or the general reciprocal of an operator.

The first part of the synthesis delineated in the foregoing paragraphs i.e. that part relating to the theory of the Langragean Multiplier technique, is supposed to be so well-known that it will suffice to refer to the bibliography.[6)]

However, since one may suppose that knowledge of the results of development and application of the theory of the generalized inverse technique is rather dispersed in spite of the rapidly increasing number of papers and books that are appearing, we think it expedient to give a short exposition of it hereafter. Being the second main part of the synthesis mentioned herebefore the generalized inverse idea forms one of the basic points in

our line of thoughts setting up a new determination model for preference structures.
As such this study can be considered as an extension of the number of non-statistical applications of the theory of generalized inverses of matrices in solving the matrix equation AX = B for the matrix X.
Besides it can deliver good examples where minimizing the so-called Euclidean Norm, as the particular choice out from the various possibilities as regards minimization of general norms, plays a non-arbitrary and meaningful role in order to get significant results.
Discussion of the concept of generalized inverses as well as literature on the same subject concern algebraical, analytical as well as numerical questions. Many of them are directly related to the concept of the 'Normal Inverse' of a matrix. Therefore in next subparagraph we start with discussion of some algebraical and analytical questions, using some topics of the linear operator theory, in order to get more insight into the relationship that exists between the concepts of the 'Normal' and 'Generalized' inverses. Ultimately we will be interested especially in developing solutions of numerical questions and want to dispose of a stable arithmetical procedure for calculating the Moore-Penrose or Pseudo-Inverse of matrix $\mathcal{A}_t$ in system (II.B.2.2.c).
This Pseudo-Inverse will be defined in next subparagraph as the unique solution of four special matrix-equations.
In the same paragraph we demonstrate this special concept of the generalized inverse is one of the various types of generalized inverses to be distinguished. Every type can be characterized listing their individual properties as regards satisfying one or more of the four matrix-equations mentioned herebefore.
Paragraph II.B.3.2 will be devoted to an arbitrary numerical example of solving a linear system of equations using the Moore-Penrose-Inverse technique. The same solution computing technique demonstrated there will be used for the D.S.I.D.-model.
The graphical presentation of the results in § II.B.3.2 is a good starting point to handle one of the subjectmatters of § II.B.4.1 i.e. those as regards justification of the ultimate choice of the Moore-Penrose Inverse in our D.S.I.D.-model.

§ II.B.3.1. Main concepts of the Generalized Inverse

The concept of Generalized Inverses is a generalization of the classical notion of the inverse or reciprocal of a non-singular, square matrix. Literature on this subject-matter shows a hierarchy of generalized inverses can be established by the use of four main definitions.
These definitions will be summarized in this subparagraph using four matrix-equations. Although all matrices are often defined over the complex number field analogous results can be obtained by our assumption that the matrices are defined over the real number field.
We accepted this procedure because the linear systems of matrices and vectors we are dealing with in our D.S.I.D.-model are always defined over the real number field.
Therefore, in the discussion about the relationship that exists between the concepts of the 'Normal' and 'Generalized' inverses of a matrix $\bar{A}_t$ as we had in system (II.B.2.2.c) considering it as a representation of a linear operator on a finite dimensional vector space, the setting of the results is assumed to be the Euclidean n-dimensional vector space over the real number field denoted by R^n instead of E^n as we did herebefore.
After this discussion we shall define the various concepts of the generalized inverse. In order to give more insight into the relationships that exist among the various concepts of the generalized inverse we will establish the numerical existence of a generalized inverse of the same arbitrary $\bar{A}_t$-matrix of a linear system of equations as we shall use in the numerical example of next subparagraph and show how the other concepts can be constructed from the first one to be defined hereafter as being the matrix $X = \bar{A}_t^g$ satisfying the first matrix-equation condition stated in II.B.3.1.h. Doing so we shall make use of some well-known theorems of the theory of linear algebra. Besides we only provide proofs of the existence of the various concepts of the generalized inverse for this numerical case. The more general proofs can be found in the bibliography quoted in the references.[7)]
Let we turn our attention now to the relationship that exists between the 'normal' and 'generalized' inverse of a matrix.

Looking at system (II.B.2.2.c) of paragraph II.B.2.2 one can consider it as a special case of the matrix-equation AX = B, where A is an (m×n)-matrix. X is an (n×1) vector and B is an (m×1) vector whereas all the elements come from the real number field.
Apart from the special $\bar{A}_t$-matrix, $\bar{p}_t$- and g_t-vectors of the system, for the sake of uniqueness of the symbols we can think again of a same linear system i.e.

$$\bar{\mathcal{A}}_t \cdot \bar{p}_t = g_t \qquad \text{(II.B.3.1.a)}$$

where $\bar{\mathcal{A}}_t$ is an (m×n)-matrix, $\bar{p}_t$ an (n×1) vector and g_t an (m×1) vector of which the elements come from the set of real numbers $\forall_t^{t=1,\ldots,T}$. In system (II.B.3.1.a) one can consider the multiplication of the (m×n)-$\bar{\mathcal{A}}_t$ matrix with an (n×1)-$\bar{p}_t$ vector as a linear transformation giving for each vector $\bar{p}_t$, being an element of the n-dimensional real vector space ($\bar{p}_t \in R^n$), a vector $g_t = \bar{\mathcal{A}}_t \cdot \bar{p}_t$ in a subspace of the n-dimensional real vectorspace denoted by R^m.
One calls R^n the 'domain' of $\bar{\mathcal{A}}_t$ and one subspace of R^m denotes the "Range" space of the linear transformation $\bar{\mathcal{A}}_t$. This "Range" is generally indicated in literature by $R(\bar{\mathcal{A}}_t)$. Besides one talks about the "Null"-space of the linear transformation $\bar{\mathcal{A}}_t$ indicated by $N(\bar{\mathcal{A}}_t)$.
This nullspace consists of all vectors $\bar{p}_t \in R^n$ satisfying the relation:

$$\bar{\mathcal{A}}_t \cdot \bar{p}_t = \underline{0} \qquad \text{(II.B.3.1.b)}$$

where $\underline{0}$ is the (m×1)-nullvector.
The 'dimension' of $R(\bar{\mathcal{A}}_t)$ is called the rank r of matrix $\bar{\mathcal{A}}_t$.
This rank equals to the number of linear independent column vectors in matrix $\bar{\mathcal{A}}_t$. In literature one defines normally the inverse of a square and non-singular matrix $\bar{\mathcal{A}}_t$ and discusses the properties of this so-called "Normal Inverse". If matrix $\bar{\mathcal{A}}_t$ has a normal inverse it must have a square

form and be non-singular i.e. the determinant of matrix $\bar{\mathscr{A}}_t$ must be unequal to zero or $|\bar{\mathscr{A}}_t| \neq 0$.

If $\bar{\mathscr{A}}_t$ is an (m×n)-matrix we are dealing with the case of existence of the normal inverse if m = n and n = r. In this case there exists only the unique solution for system (II.B.3.1.a) which can be obtained by

$$\bar{p}_t = \bar{\mathscr{A}}_t^{-1} \cdot g_t, \text{ where} \tag{II.B.3.1.c}$$

$\bar{\mathscr{A}}_t^{-1}$ is the normal inverse of matrix $\bar{\mathscr{A}}_t$.

However there are many circumstances in which $\bar{\mathscr{A}}_t$ is rectangular or square and singular. We are dealing with such a situation in our original system (II.B.2.2.c).

In these cases the theory of the 'Generalized Inverses' still enables us to obtain solutions (or approximations of them). Actually a "Normal Inverse" of the matrix $\bar{\mathscr{A}}_t$ does not always exist whereas a 'Generalized Inverse' of $\bar{\mathscr{A}}_t$ does indeed. This circumstance will enable us ultimately to derive later on 'overall' preferences structures as yet undefined. As regards the relationship between the 'Normal Inverse' and the 'Generalized Inverse' of the matrix $\bar{\mathscr{A}}_t$ it will be sufficient to expose the relationship between the 'Normal Inverse' and a special concept of the generalized inverse i.e. that of the Pseudo-Inverse because the relationships that exist between the Pseudo-Inverse and the other concepts of the generalized inverse are exposed hereafter so the relationships between the latter and the 'Normal Inverse' become automatically clear.

If we have an (n×n)-matrix $\bar{\mathscr{A}}_t$ in system (II.B.3.1.a) which is singular ($r < n$) and if the nullspace $N(\bar{\mathscr{A}}_t)$ contains also non-null-vectors we can say for any known vector $g_t \in R^m$ ($m = n$) analogous with case I and case II of § II.B.2.3:

<u>Case I'</u>: There exists an infinite number of solutions because we can add to any solution a non-null-vector of $N(\mathcal{A}_t)$ and thus derive another solution (Case of Consistency of system (II.B.3.1.a)).

<u>Case II'</u>: There are no solutions i.e. $g_t \notin R(\mathcal{A}_t)$ (Case of Inconsistency of system (II.B.3.1.a)).

In the latter case II' we do not require the equality between $\mathcal{A}_t \cdot \bar{p}_t$ and g_t in system (II.B.3.1.a).

As we saw already in Case II of § II.B.2.3 if the matrix $\mathcal{A}_t$ is known (possibly singular) we can try to find for any $g_t \in R^m$ that vector $\bar{p}_{t,\hat{o}}$ which minimizes the sum of squares of deviation between $\mathcal{A}_t \cdot \bar{p}_t$ and g_t. In the literature it is proved this vector $\bar{p}_{t,\hat{o}}$ is found by using the Pseudo-Inverse of matrix $\mathcal{A}_t$ i.e.

$$\bar{p}_{t,\hat{o}} = \mathcal{A}_t^{+} g_t \qquad \text{(II.B.3.1.d)}$$

For $\bar{p}_{t,\hat{o}}$ of system (II.B.3.1.d) we can give the following geometrical interpretation:

Let us project orthogonally g_t on $R(\mathcal{A}_t)$ in such a way that $g_{t,p}$ is this projection i.e.

$$g_{t,p} \in R(\mathcal{A}_t).$$

One can proof that there is only one vector in $R(\mathcal{A}_t')$ i.e. the vector $\bar{p}_{t,\hat{o}}$ for which $\mathcal{A}_t \bar{p}_{t,\hat{o}} = g_{t,p}$. $R(\mathcal{A}_t')$ denotes the "Range" space of the linear transformation of the matrix $\mathcal{A}_t'$, where $\mathcal{A}_t'$ is the transpose of matrix $\mathcal{A}_t$. It can also be proved that:

$$\|\bar{\mathscr{A}}_t\ \bar{\underline{p}}_{t,\hat{o}} - \underline{g}_t\|^2 = \min_{\bar{\underline{p}}_t}\|\bar{\mathscr{A}}_t\ \bar{\underline{p}}_t - \underline{g}_t\|^2,$$

where $\bar{\underline{p}}_t \in R^n$.

The relations between the solution $\bar{\underline{p}}_{t,\hat{o}}$ of the modified problem (being the "Best Approximate Solution" of the original problem (II.B.3.1.a) in the case of inconsistency) and $\underline{g}_t$ are linear and denoted by system (II.B.3.1.d).

Dealing with case I' of consistency of system (II.B.3.1.a) i.e. $\underline{g}_t \in R(\bar{\mathscr{A}}_t)$ we can say that $\underline{g}_t$ can be expressed as a linear combination of r 'basis'-vectors. The nullspace $N(\bar{\mathscr{A}}_t)$ of the linear transformation $\mathscr{A}_t$ consists of all vectors $\bar{\underline{p}}_t \in R^n$ which satisfy relation (II.B.3.1.b) and has dimension (n-r). This dimension is called the 'Nullity' of $\bar{\mathscr{A}}_t$ and means that $N(\bar{\mathscr{A}}_t)$ is spanned by (n-r) linear independent vectors.

The set of these (n-r) column vectors form a 'basis' of $N(\bar{\mathscr{A}}_t)$ and we shall denote it by $\bar{\mathscr{A}}_t^B$. Thus all the vectors in $N(\bar{\mathscr{A}}_t)$ can be written as $\bar{\mathscr{A}}_t^B.\underline{s}$, where $\underline{s}$ is an arbitrary (n-r)-real columnvector.
We can give now the following geometrical interpretation used in the linear operator theory:

$N(\bar{\mathscr{A}}_t)$ and $R(\bar{\mathscr{A}}_t')$ are the orthogonal complements of each other i.e. the vectors $\bar{\underline{p}}_{t,u} \in R(\bar{\mathscr{A}}_t')$ and $\bar{\underline{p}}_{t,N} \in N(\bar{\mathscr{A}}_t)$ are perpendicular in the Euclidean space R^n and $\bar{\underline{p}}_{t,o} = \bar{\underline{p}}_{t,u} + \bar{\underline{p}}_{t,N}$ denotes the orthogonal decomposition of $\bar{\underline{p}}_{t,o}$ in the Euclidean space R^n.
In the literature one has proved that the Pseudo-Inverse of the Matrix $\bar{\mathscr{A}}_t$ i.e. $\bar{\mathscr{A}}_t^+$ transforms vectors of R^m into vectors being elements of $R(\bar{\mathscr{A}}_t')$, where $R(\bar{\mathscr{A}}_t')$ is a subspace of R^n in the following sense:

$$- \forall_{\bar{\underline{p}}_{t,u}}\ \bar{\underline{p}}_{t,u} \in R(\bar{\mathscr{A}}_t') \qquad \text{one can find } \bar{\mathscr{A}}_t^+\ \bar{\mathscr{A}}_t.\bar{\underline{p}}_{t,u} = \bar{\underline{p}}_{t,u}$$

- $\forall_{g_t}^{g_t} \in R(\bar{\mathcal{A}}_t)$ one obtains $\bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^{+}\cdot g_t = g_t$

Concluding this first discussion we remark:

If the null space $N(\bar{\mathcal{A}}_t)$ has dimension zero i.c. $(n-r) = 0$ and $n = m$ the solution of system (II.B.3.1.a) can be given as:

$$\bar{p}_{t,o} = \bar{\mathcal{A}}_t^{+}\cdot g_t + \bar{\mathcal{A}}_t^{B}\cdot \underline{s} = \bar{\mathcal{A}}_t^{-1}\cdot g_t = \bar{p}_{t,u}$$

If $N(\bar{\mathcal{A}}_t)$ has dimension $(n-r) > 0$ (or $r < n$) there exists a range of solutions given by

$$\bar{p}_{t,o} = \bar{\mathcal{A}}_t^{+}\ g_t + \bar{\mathcal{A}}_t^{B}\cdot \underline{s} \qquad \text{(II.B.3.1.e)}$$

From system (II.B.3.1.e) we derive:

$$\bar{\mathcal{A}}_t\ \bar{p}_{t,o} = \bar{\mathcal{A}}_t\bar{\mathcal{A}}_t^{+}\ g_t + \bar{\mathcal{A}}_t\bar{\mathcal{A}}_t^{B}\ \underline{s} = g_t + 0 = g_t$$

since $\bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^{B} = 0$ (the null-matrix) and $\bar{\mathcal{A}}_t\bar{\mathcal{A}}_t^{+}\cdot g_t = g_t$ as we saw herefore. System (II.B.3.1.e) is equivalent with system (II.B.2.3.a), however whereas $\bar{\mathcal{A}}_t^{B}$ in the first system is a 'basis' of $N(A_t)$, $(I-\bar{A}_t^{+}\ \bar{A}_t)$ in the latter system consists now of n linear dependent n-dimensional column vectors spanning the null space $N(\bar{\mathcal{A}}_t)$ so we can write:

$$\{\bar{\mathcal{A}}_t^{B}\ \underline{s}/\underline{s} \in R^{n-r}\} \qquad \text{(II.B.3.1.f)}$$

and

$$\{(\mathcal{I} - \bar{\mathcal{A}}_t^{+}\ \bar{\mathcal{A}}_t)\underline{r}/\underline{r} \in R^{n}\}$$

From (II.B.3.1.f) it will be clear that for case I' of consistency of system (II.B.3.1.a) the formulation of the general solution analogous with (II.B.2.3.a) must be preferred i.e.

$$\bar{\underline{p}}_{t,o} = \bar{\mathscr{A}}_t^{\,+}\, \underline{q}_t + (I - \bar{\mathscr{A}}_t^{\,+}\, \bar{\mathscr{A}}_t)\,\underline{r} \qquad \text{(II.B.3.1.g)}$$

because if the $\bar{\mathscr{A}}_t$-matrix and $\underline{q}_t$-vector in system (II.B.3.1.a) are known the general solution of this system can be formulated immediately by use of (II.B.3.1.g) if we have the suitable arithmatical procedure for calculating $\bar{\mathscr{A}}_t^{\,+}$. Knowledge of the $\bar{\mathscr{A}}_t$-matrix will be unsufficient many times for formulating immediately $\bar{\mathscr{A}}_t^{\,B}$ of system (II.B.3.1.e).

After the foregoing discussion we can turn now our attention to the definition of the main concept of the 'Generalized Inverse' and show in a numerical example how the Pseudo-Inverse $\bar{\mathscr{A}}_t^{\,+}$, we used already, can be derived from the other concepts of the generalized inverse to be distinguished.

(In next subparagraph it will become clear in what way solutions of a system as (II.B.3.1.a) using other concepts of the Generalized Inverse in the case of consistency are related with the 'General Solution' formulation (II.B.3.1.g) using the Moore-Penrose Inverse technique.)

Let $M_{m,n}$ be the set of matrices consisting of m rows and n columns of which the elements come from the set of real numbers.

Let $\bar{\mathscr{A}}_t$ be again an (m×n)-matrix of system (II.B.3.1.a) and being also an element of $M_{m,n}$.

The following matrix-equation conditions can be used to define the main concepts of the Generalized Inverse of the matrix $\bar{\mathscr{A}}_t$:

II.B.3.1.h: $\bar{\mathscr{A}}_t . X . \bar{\mathscr{A}}_t = \bar{\mathscr{A}}_t$

II.B.3.1.i: $X . \bar{\mathscr{A}}_t . X = X$

II.B.3.1.j: $(X . \bar{\mathscr{A}}_t)^* = X . \bar{\mathscr{A}}_t$

II.B.3.1.k: $(\bar{\mathscr{A}}_t . X)^* = \bar{\mathscr{A}}_t . X$

where $(\quad)^*$ denotes the transpose of the matrix.

Definition II.B.3.1.h': A generalized inverse of the matrix $\mathcal{A}_t$ is a matrix $X = \mathcal{A}_t{}^g$ satisfying condition II.B.3.1.h.

Definition II.B.3.1.i': A reflexive generalized inverse of the matrix $\mathcal{A}_t$ is a matrix $X = \mathcal{A}_t{}^r$ satisfying conditions II.B.3.1.h and II.B.3.1.i.

Definition II.B.3.1.j': A left weak generalized inverse of the matrix $\mathcal{A}_t$ is a matrix $X = \mathcal{A}_t{}^{lw}$ satisfying conditions II.B.3.1.h and II.B.3.1.i and II.B.3.1.j.

Definition II.B.3.1.j": A right weak generalized inverse of the matrix $\mathcal{A}_t$ is a matrix $X = \mathcal{A}_t{}^{rw}$ satisfying conditions II.B.3.1.h and II.B.3.1.i and II.B.3.1.k.

Definition II.B.3.1.k': A pseudo-inverse or Moore-Penrose inverse of the matrix $\mathcal{A}_t$ is a matrix $X = \mathcal{A}_t{}^+$ satisfying all four matrix equation conditions i.e. II.B.3.1.h, II.B.3.1.i, II.B.3.1.j and II.B.31.k.

Making use of conditions II.B.3.1.h/II.B.3.1.k, of definitions II.B.3.1.h'/II.B.3.1.k' and of some well-known theorems of the theory of linear algebra we can derive hereafter some important theorems as regards the various main concepts of a generalized inverse of a matrix $\mathcal{A}_t$ of system (II.B.3.1.a). Besides it will become clear which inclusion relationships there exist among the sets of these concept whereas the uniqueness of the Pseudo-Inverse concept will be established. Every theorem is applicated lateron where we establish the numerical existence of the different concepts of a generalized inverse for a certain numerically specified $\mathcal{A}_t$-matrix.

We restate the following well-known theorems of the theory of linear algebra:

(a) Any rectangular matrix $\mathcal{A}_t$ of rank r is equivalent to a matrix of the form:

$$B = \left[\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right] \tag{II.B.3.1.1}$$

where E is an r×r-non-singular matrix and 0's are null matrices, or of the form

$$B_1 = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \tag{II.B.3.1.1'}$$

where I_r is the r^{th}-order unity matrix and 0's denote again null-matrices.

The matrices B and B_1 are obtained by equivalence transformation on matrix $\mathcal{A}_t$ consisting of a series of elementary row and column operations i.e.:

$$B = P \mathcal{A}_t Q$$

where P and Q are nonsingular matrices characterizing the series of elementary row respectively the series of elementary column operations,
and

$$B_1 = P_1 \mathcal{A}_t Q_1$$

where P_1 and Q_1 are again non-singular matrices.

Depending on the size of $\mathcal{A}_t$, some or all of the 0 submatrices in the right-hand sides of (II.B.3.1.1) and (II.B.3.1.1') may not appear.

(b) If $\mathcal{A}_t$ is a square matrix with all different eigenvalues similarity transformation on this matrix can result into diagonalization of it i.e.

$$B_2 = P_2^{-1} \mathcal{A}_t P_2 \qquad \text{(II.B.3.1.m)}$$

where B_2 is the diagonal matrix whose diagonal elements are the eigenvalues of $\mathcal{A}_t$ and P_2 is a non-singular matrix consisting of linearly independent eigenvectors corresponding to the different eigenvalues of matrix $\mathcal{A}_t$. If $\mathcal{A}_t$ is not symmetric the matrix P_2 is not in general an orthogonal matrix, where a matrix P_2 is called orthogonal if its inverse is its transpose i.e. $P_2^{-1} = P_2^*$.

(c) If $\mathcal{A}_t$ is a symmetric matrix it can be diagonalized by an orthogonal similarity transformation i.e.:

$$B_3 = P_3^{-1} \mathcal{A}_t P_3 = P_3^* \mathcal{A}_t P_3 \qquad \text{(II.B.3.1.n)}$$

where B_3 is the diagonal matrix whose diagonal elements are the eigenvalues of $\mathcal{A}_t$ whereas the matrix P_3 which is used to diagonalize $\mathcal{A}_t$ has as its columns an orthonormal set of eigenvectors for $\mathcal{A}_t$.

Let the matrix $\mathcal{A}_t$ of system (II.B.3.1.a) be numerically specified as follows:

$$\mathcal{A}_t = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

and the vector g_t of the same system as:

$$g_t = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Substituting them into system (II.B.3.1.a) we get the linear system:

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} . \begin{bmatrix} \bar{p}_{t,1} \\ \bar{p}_{t,2} \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad \text{(II.B.3.1.o)}$$

$$\bar{\mathcal{A}}_t \quad . \quad \bar{p}_t \quad = \quad g_t$$

Before we establish the numerical existence of the different concepts of a generalized inverse for $\bar{\mathcal{A}}_t$ in (II.B.3.1.o) we shall derive first some important theorems as regards these main concepts for an arbitrary matrix $\bar{\mathcal{A}}_t$ of system (II.B.3.1.a).

From condition II.B.3.1.h and definition II.B.3.1.h' and using theorem (a) we derive:

$$\bar{\mathcal{A}}_t \; X \; \bar{\mathcal{A}}_t = \bar{\mathcal{A}}_t$$

$$P^{-1} \; B \; Q^{-1} \; X \; P^{-1} \; B \; Q^{-1} = \bar{\mathcal{A}}_t \Rightarrow$$

$$B \; Q^{-1} \; X \; P^{-1} \; B = P \; \bar{\mathcal{A}}_t \; Q = B \Rightarrow$$

$$Q^{-1} \; X \; P^{-1} = B^g$$

$$X = Q \; B^g \; P \qquad \text{(II.B.3.1.p)}$$

where B^g is a generalized inverse of matrix B corresponding with definition II.B.3.1.h' i.e.

$B \; B^g \; B = B$ or using (II.B.3.1.1)

$$\left[\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right] . \begin{bmatrix} B^g \end{bmatrix} = \left[\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} B^g \end{bmatrix} = \left[\begin{array}{c|c} E^{-1} & \alpha \\ \hline \beta & \gamma \end{array}\right]$$

where α, β and γ are arbitrary matrices and E^{-1} is the normal inverse of matrix E of system (II.B.3.1.1).
Instead of (II.B.3.1.p) we can write now:

$$X = Q\, B^{g}\, P = \begin{bmatrix} Q \end{bmatrix} \cdot \left[\begin{array}{c|c} E^{-1} & \alpha \\ \hline \beta & \gamma \end{array}\right] \cdot \begin{bmatrix} P \end{bmatrix}$$

Substituting the latter matrix X into matrix condition II.B.3.1.h it is easily verified that:

$$\mathcal{A}_t\, X\, \mathcal{A}_t = P^{-1}\, B\, Q^{-1}\, Q\, B^{g}\, P\, P^{-1}\, B\, Q^{-1} = P^{-1}\, B\, B^{g}\, B\, Q^{-1} =$$

$$P^{-1}\, B\, Q^{-1} = \mathcal{A}_t$$

From definition II.B.3.1.h' we get:

<u>Theorem II.B.3.1.h"</u>: For matrix $\mathcal{A}_t$ of system (II.B.3.1.a) there exists a generalized inverse $\mathcal{A}_t^{\,g}$ which can be found as:

$$\mathcal{A}_t^{\,g} = Q\, B^{g}\, P \qquad \text{(II.B.3.1.q)}$$

From conditions II.B.3.1.h, II.B 3.1.i and definition II.B.3.1.i' and using theorem II.B.3.1.h" we derive:

(1)
$$\left.\begin{array}{l} \mathcal{A}_t\, X\, \mathcal{A}_t = \mathcal{A}_t \\ \\ X\, \mathcal{A}_t\, X = X \end{array}\right] \Rightarrow \mathcal{A}_t(X\, \mathcal{A}_t\, X)\mathcal{A}_t = \mathcal{A}_t \Rightarrow$$

$$\Rightarrow X\, \mathcal{A}_t\, X = \mathcal{A}_t^{g_1}\mathcal{A}_t\, \mathcal{A}_t^{g_2} = \mathcal{A}_t^{g}$$

(2)
$$(X\, \mathcal{A}_t\, X)\mathcal{A}_t(X\, \mathcal{A}_t X) = X\, \mathcal{A}_t\, X$$

From (1) and (2) it is easily verified that:

$$X\, \mathcal{A}_t\, X = \mathcal{A}_t^{g_1}\, \mathcal{A}_t\, \mathcal{A}_t^{g_2} = \mathcal{A}_t^{\,r} \text{ or}^{8)}$$

<u>Theorem II.B.3.1.i"</u>: For matrix $\bar{\mathcal{A}}_t$ of system (II.B.3.1.a) there exists a reflexive generalized inverse $\bar{\mathcal{A}}_t^{\,r}$ which can be found as:

$$\bar{\mathcal{A}}_t^{\,r} = \bar{\mathcal{A}}_t^{g_1}\ \bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^{g_2}$$

where $\bar{\mathcal{A}}_t^{g_1}$ and $\bar{\mathcal{A}}_t^{g_2}$ are generalized inverses of matrix $\bar{\mathcal{A}}_t$ corresponding with definition II.B.3.1.h'. Using theorem II.B.3.1.h" we can write now:

$$\bar{\mathcal{A}}_t^{\,r} = (Q\ B^{g_1}\ P)(P^{-1}\ B\ Q^{-1})(Q\ B^{g_2}\ P) = Q\ B^{g_1}\ B\ B^{g_2}\ P \qquad \text{(II.B.3.1.r)}$$

where B^{g_1} and B^{g_2} are generalized inverses of matrix B.

From conditions II.B.3.1.h, II.B.3.1.i, II.B.3.1.j and definition II.B.3.1.j' and using the theorems II.B.3.1.h" and II.B.3.1.i" we derive:

$$\left.\begin{array}{l} \left.\begin{array}{l} \bar{\mathcal{A}}_t\ X\ \bar{\mathcal{A}}_t = \bar{\mathcal{A}}_t \\[2ex] X\ \bar{\mathcal{A}}_t\ X = X \end{array}\right] \Rightarrow \begin{array}{l} (X\ \bar{\mathcal{A}}_t)^*\ \bar{\mathcal{A}}_t^{\,*} = \bar{\mathcal{A}}_t^{\,*} \\[2ex] X^*(X\ \bar{\mathcal{A}}_t)^* = X^* \end{array} \\[4ex] (X\ \bar{\mathcal{A}}_t)^* = X\ \bar{\mathcal{A}}_t \end{array}\right] \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow X\ \bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^{\,*} = \bar{\mathcal{A}}_t^{\,*} \\[2ex] \quad X^*\ X\ \bar{\mathcal{A}}_t = X^* \Rightarrow \bar{\mathcal{A}}_t^{\,*}\ X^*\ X = X \end{array}\right] \Rightarrow$$

$$\Rightarrow \bar{\mathcal{A}}_t^{\,*}\ X^*\ X\ \bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^{\,*} = \bar{\mathcal{A}}_t^{\,*} \Rightarrow \bar{\mathcal{A}}_t\bar{\mathcal{A}}_t^{\,*}\ X^*\ X\ \bar{\mathcal{A}}_t\bar{\mathcal{A}}_t^{\,*} = \bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^{\,*} \Rightarrow$$

$$\left.\begin{array}{l} X^*\ X = (\bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^{\,*})^g \\ \text{and} \\ \bar{\mathcal{A}}_t^{\,*}\ X^*\ X = \bar{\mathcal{A}}_t^{\,g} \end{array}\right] \Rightarrow \bar{\mathcal{A}}_t^{\,g} = \bar{\mathcal{A}}_t^{\,*}(\bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^{\,*})^g$$

Using theorem II.B.3.1.i" we derive:

$$\mathcal{A}_t^{\,r} = \mathcal{A}_t^{g_1}\,\mathcal{A}_t\,\mathcal{A}_t^{g_2} = \mathcal{A}_t^{*}(\mathcal{A}_t\,\mathcal{A}_t^{*})^{g_1}\,\mathcal{A}_t\,\mathcal{A}_t^{*}(\mathcal{A}_t\,\mathcal{A}_t^{*})^{g_2} \Rightarrow$$

$$\Rightarrow \mathcal{A}_t^{\,r} = \mathcal{A}_t^{*}(\mathcal{A}_t\,\mathcal{A}_t^{*})^{r}$$

because $(\mathcal{A}_t\,\mathcal{A}_t^{*})^{g_1}\,\mathcal{A}_t\,\mathcal{A}_t^{*}(\mathcal{A}_t\,\mathcal{A}_t^{*})^{g_2} = (\mathcal{A}_t\,\mathcal{A}_t^{*})^{r}$.

It is easily verified that $(\mathcal{A}_t\,\mathcal{A}_t^{*})^{r}$ and $\mathcal{A}_t^{*}(\mathcal{A}_t\mathcal{A}_t^{*})^{r}\,\mathcal{A}_t$ are symmetric. By this it is true that:

<u>Theorem II.B.3.1.j'"</u>: For matrix $\mathcal{A}_t$ of system (II.B.3.1.a) there exists a left weak generalized inverse $\overset{1}{\mathcal{A}}{}^{w}$ which can be found as:

$$\overset{1}{\mathcal{A}}{}_t^{w} = \mathcal{A}_t^{*}(\mathcal{A}_t\,\mathcal{A}_t^{*})^{r}$$

Using theorem (c) a singular value decomposition of matrix $\mathcal{A}_t$ [9] can be denoted by the eigenvalue decomposition of the symmetric matrix $\mathcal{A}_t\mathcal{A}_t^{*}$ i.e.

$$B_3 = P_3^{-1}(\mathcal{A}_t\,\mathcal{A}_t^{*})P_3 = P_3^{*}(\mathcal{A}_t\,\mathcal{A}_t^{*})P_3$$

where B_3 is the diagonal matrix whose diagonal elements are the eigenvalues of $\mathcal{A}_t\,\mathcal{A}_t^{*}$ whereas matrix P_3 has as its columns an orthogonal set of eigenvectors for $\mathcal{A}_t\,\mathcal{A}_t^{*}$.

Analogous with derivation of $\mathcal{A}_t^{g}$ it is easily verified that:

$$(\mathcal{A}_t\,\mathcal{A}_t^{*})^{g} = P_3\,B_3^{g}\,P_3^{*}$$

where B_3^{g} is a generalized inverse of B_3 or using definition II.B.3.1.h'

$$B_3\,B_3^{g}\,B_3 = B_3 \text{ or}$$

$$\left[\begin{array}{c|c} E_3 & 0 \\ \hline 0 & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} E_3^{-1} & \alpha_3 \\ \hline \beta_3 & \gamma_3 \end{array}\right] \cdot \left[\begin{array}{c|c} E_3 & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} E_3 & 0 \\ \hline 0 & 0 \end{array}\right]$$

where E_3^{-1} is the Normal inverse of E_3 the latter being the diagonal matrix whose diagonal elements are the non-null eigenvalues of $\mathcal{A}_t\,\mathcal{A}_t^*$ whereas α_3, β_3 and γ_3 are again arbitrary matrices.[10)]

We can now write:

$$\bar{\mathcal{A}}_t^{1w} = \bar{\mathcal{A}}_t^*(\bar{\mathcal{A}}_t\,\bar{\mathcal{A}}_t^*)^r = \bar{\mathcal{A}}_t^*(\bar{\mathcal{A}}_t\,\bar{\mathcal{A}}_t^*)^{g_1}\,\bar{\mathcal{A}}_t\,\bar{\mathcal{A}}_t^*(\bar{\mathcal{A}}_t\,\bar{\mathcal{A}}_t^*)^{g_2} =$$

$$(Q^{-1})^*\ B^*(P^{-1})^*\ P_3\ B_3^{g_1}\ P_3^*\ P_3\ B_3\ P_3^*\ P_3\ B_3^{g_2}\ P_3^* =$$

$$(Q^{-1})^*\ B^*(P^{-1})^*\ P_3\ B_3^{g_1}\ B_3\ B_3^{g_2}\ P_3^* \qquad \text{(II.B.3.1.s)}$$

where $B_3^{g_1}$ and $B_3^{g_2}$ are generalized inverses of matrix B_3.

From conditions II.B.3.1.h, II.B.3.1.i, II.B.3.1.k and definition II.B.3.1.j" and using theorems II.B.3.1.h" and II.B.3.1.i" we derive:

$$\left.\begin{array}{l} \left.\begin{array}{l} \bar{\mathcal{A}}_t\ X\ \bar{\mathcal{A}}_t = \bar{\mathcal{A}}_t \\[2ex] X\ \bar{\mathcal{A}}_t\ X = X \end{array}\right] \Rightarrow \begin{array}{l} \bar{\mathcal{A}}_t^*(\bar{\mathcal{A}}_t\ X)^* = \bar{\mathcal{A}}_t^* \\[2ex] (\bar{\mathcal{A}}_t\ X)^*\ X^* = X^* \end{array} \\[2ex] (\bar{\mathcal{A}}_t\ X)^* = \bar{\mathcal{A}}_t\ X \end{array}\right] \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \bar{\mathcal{A}}_t^*\ \bar{\mathcal{A}}_t\ X = \bar{\mathcal{A}}_t^* \\[2ex] \quad \mathcal{A}_t\ X\ X^* = X^* \Rightarrow X\ X^*\ \bar{\mathcal{A}}_t^* = X \end{array}\right] \Rightarrow$$

$$\Rightarrow \bar{\mathcal{A}}_t^*\ \bar{\mathcal{A}}_t\ X\ X^*\ \bar{\mathcal{A}}_t^* = \bar{\mathcal{A}}_t^* \Rightarrow \bar{\mathcal{A}}_t^*\ \bar{\mathcal{A}}_t\ X\ X^*\ \bar{\mathcal{A}}_t^*\ \bar{\mathcal{A}}_t = \bar{\mathcal{A}}_t^*\ \bar{\mathcal{A}}_t \Rightarrow$$

$$\left.\begin{array}{l} X\,X^* = (\mathcal{A}_t^*\,\mathcal{A}_t)^g \\ \text{and} \\ X\,X^*\,\mathcal{A}_t^* = \mathcal{A}_t^g \end{array}\right\} \Rightarrow \mathcal{A}_t^g = (\mathcal{A}_t^*\,\mathcal{A}_t)^g\,\mathcal{A}_t^*$$

using theorem II.B.3.1.i" we derive:

$$\mathcal{A}_t^r = \mathcal{A}_t^{g_1}\,\mathcal{A}_t\,\mathcal{A}_t^{g_2} = (\mathcal{A}_t^*\,\mathcal{A}_t)^{g_1}\,\mathcal{A}_t^*\,\mathcal{A}_t(\mathcal{A}_t^*\,\mathcal{A}_t)^{g_2}\,\mathcal{A}_t^* \Rightarrow$$

$$\Rightarrow \mathcal{A}_t^r = (\mathcal{A}_t^*\,\mathcal{A}_t)^r\,\mathcal{A}_t^*$$

because $(\mathcal{A}_t^*\,\mathcal{A}_t)^{g_1}\,\mathcal{A}_t^*\,\mathcal{A}_t(\mathcal{A}_t^*\,\mathcal{A}_t)^{g_2} = (\mathcal{A}_t^*\,\mathcal{A}_t)^r$

It is easily verified $(\mathcal{A}_t^*\,\mathcal{A}_t)^r$ is symmetric and thus

$$(\mathcal{A}_t\,\mathcal{A}_t^*)^r\,\mathcal{A}_t^*\,\mathcal{A}_t \text{ is symmetric.}$$

By this it is true that:

<u>Theorem II.B.3.1.j""</u>: For matrix $\mathcal{A}_t$ of system (II.B.3.1.a) there exists a right weak-generalized inverse $\overset{r}{\mathcal{A}}{}_t^{w}$ which can be found as:

$$\overset{r}{\mathcal{A}}{}_t^{w} = (\mathcal{A}_t^*\,\mathcal{A}_t)^r\,\mathcal{A}_t^*$$

Using theorem (c) a singular value decomposition of matrix $\mathcal{A}_t$ can be denoted by the eigenvalue decomposition of the symmetric matrix $\mathcal{A}_t^*\,\mathcal{A}_t$ i.e.

$$B_4 = P_4^{-1}(\mathcal{A}_t^*\,\mathcal{A}_t)P_4 = P_4^*(\mathcal{A}_t^*\,\mathcal{A}_t)P_4$$

where B_4 is the diagonal matrix whose diagonal elements are the eigenvalues of $\mathcal{A}_t^*\,\mathcal{A}_t$ whereas matrix P_4 has as its columns an orthonormal set of eigenvectors for $\mathcal{A}_t^*\,\mathcal{A}_t$.

Analogous with derivation of $\bar{\mathcal{A}}_t^g$ it is easily verified that:

$$(\bar{\mathcal{A}}_t^* \bar{\mathcal{A}}_t)^g = P_4 B_4^g P_4^*$$

where B_4^g is a generalized inverse of B_4 or using definition II.B.3.1.h':

$$B_4 B_4^g B_4 = B_4 \text{ or}$$

$$\left[\begin{array}{c|c} E_4 & 0 \\ \hline 0 & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} E_4^{-1} & \alpha_4 \\ \hline \beta_4 & \gamma_4 \end{array}\right] \cdot \left[\begin{array}{c|c} E_4 & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} E_4 & 0 \\ \hline 0 & 0 \end{array}\right]$$

where E_4^{-1} is the normal inverse of E_4 the latter being the diagonal matrix whose diagonal elements are the non-null eigenvalues of $\bar{\mathcal{A}}_t^* \bar{\mathcal{A}}_t$ whereas α_4, β_4 and γ_4 are arbitrary matrices.[11)]
We can now write:

$$\bar{\mathcal{A}}_t^{r_w} = (\bar{\mathcal{A}}_t^* \bar{\mathcal{A}}_t)^r \bar{\mathcal{A}}_t^* = (\bar{\mathcal{A}}_t^* \bar{\mathcal{A}}_t)^{g_1} \bar{\mathcal{A}}_t^* \bar{\mathcal{A}}_t (\bar{\mathcal{A}}_t^* \bar{\mathcal{A}}_t)^{g_2} \bar{\mathcal{A}}_t^* =$$

$$P_4 B_4^{g_1} P_4^* P_4 B_4 P_4^* P_4 B_4^{g_2} P_4^* (Q^{-1})^* B^* (P^{-1})^* =$$

$$P_4 B_4^{g_1} B_4 B_4^{g_2} P_4^* (Q^{-1})^* B^* (P^{-1})^* \qquad \text{(II.B.3.1.t)}$$

where $B_4^{g_1}$ and $B_4^{g_2}$ are generalized inverses of matrix B_4.

From conditions II.B.3.1.h, II.B.3.1.i, II.B.3.1.j, II.B.3.1.k, and definition II.B.3.1.k' we derive:

(1) using theorem II.B.3.1.j'":

$$\left.\begin{array}{l} \bar{\mathcal{A}}_t \; X \; \bar{\mathcal{A}}_t = \bar{\mathcal{A}}_t \\ X \; \bar{\mathcal{A}}_t \; X = X \\ (X \; \bar{\mathcal{A}}_t)^* = X \; \bar{\mathcal{A}}_t \end{array}\right\} \Rightarrow X = \overset{l}{\bar{\mathcal{A}}_t}{}^{w}$$

(2) using theorem II.B.3.1.j"":

$$\left.\begin{array}{l} \bar{\mathcal{A}}_t \; X \; \bar{\mathcal{A}}_t = \bar{\mathcal{A}}_t \\ X \; \bar{\mathcal{A}}_t \; X = X \\ (\bar{\mathcal{A}}_t \; X)^* = \bar{\mathcal{A}}_t \; X \end{array}\right\} \Rightarrow X = \overset{r}{\bar{\mathcal{A}}_t}{}^{w}$$

From theorem II.B.3.1.i" we know:

$$\bar{\mathcal{A}}_t{}^{r} = \overset{l}{\bar{\mathcal{A}}_t}{}^{w} \; \bar{\mathcal{A}}_t \; \overset{r}{\bar{\mathcal{A}}_t}{}^{w}$$

$$\bar{\mathcal{A}}_t{}^{r} = \overset{r}{\bar{\mathcal{A}}_t}{}^{w} \; \bar{\mathcal{A}}_t \; \overset{l}{\bar{\mathcal{A}}_t}{}^{w}$$

From the theorems II.B.3.1.j'" and II.B.3.1.j"" we know:

$$\overset{l}{\bar{\mathcal{A}}_t}{}^{w} = \bar{\mathcal{A}}_t{}^{*}(\bar{\mathcal{A}}_t \; \bar{\mathcal{A}}_t{}^{*})^{r}$$

and

$$\overset{r}{\bar{\mathcal{A}}_t}{}^{w} = (\bar{\mathcal{A}}_t{}^{*} \; \bar{\mathcal{A}}_t)^{r} \; \bar{\mathcal{A}}_t{}^{*}$$

So it is easily verified that:

$$\bar{\mathcal{A}}_t{}^{r} = \bar{\mathcal{A}}_t{}^{*}(\bar{\mathcal{A}}_t \; \bar{\mathcal{A}}_t{}^{*})^{r} \; \bar{\mathcal{A}}_t(\bar{\mathcal{A}}_t{}^{*} \; \bar{\mathcal{A}}_t)^{r} \; \bar{\mathcal{A}}_t{}^{*} \tag{A}$$

and

$$\bar{\mathcal{A}}_t{}^{r} = (\bar{\mathcal{A}}_t{}^{*} \; \bar{\mathcal{A}}_t)^{r} \; \bar{\mathcal{A}}_t{}^{*} \; \bar{\mathcal{A}}_t \; \bar{\mathcal{A}}_t{}^{*}(\bar{\mathcal{A}}_t \; \bar{\mathcal{A}}_t{}^{*})^{r} \tag{B}$$

It is easily verified for $X = \bar{\mathscr{A}}_t^{\,r}$ of expression (A) conditions II.B.3.1.j and II.B.3.1.k are satisfied whereas for $X = \bar{\mathscr{A}}_t^{\,r}$ of expression (B) condition II.B.3.1.j is also satisfied; however this is not true as regards condition II.B.3.1.k.
So we can say:

Theorem II.B.3.1.k": For matrix $\bar{\mathscr{A}}_t$ of system (II.B.3.1.a) there exists a pseudo-inverse $\bar{\mathscr{A}}_t^{\,+}$ which can be found as:

$$\bar{\mathscr{A}}_t^{\,+} = \overset{l}{\bar{\mathscr{A}}}{}_t^{\,w}\, \bar{\mathscr{A}}_t\, \overset{r}{\bar{\mathscr{A}}}{}_t^{\,w}$$

making use of systems (II.B.3.1.s) and (II.B.3.1.t) we can write:

$$\bar{\mathscr{A}}_t^{\,+} = (Q^{-1})^{*}\, B^{*} (P^{-1})^{*}\, P_3\, B_3^{g_1}\, B_3\, B_3^{g_2}\, P_3^{*}\, P^{-1}\, B\, Q^{-1}\, P_4\, B_4^{g_1}\, B_4\, B_4^{g_2} \cdot P_4^{*} (Q^{-1})^{*}\, B^{*} (P^{-1})^{*} \qquad \text{(II.B.3.1.u)}$$

Remarks:

(1) It is easily to verify that in general $\bar{\mathscr{A}}_t^{g}$, $\bar{\mathscr{A}}_t^{r}$, $\overset{l}{\bar{\mathscr{A}}}{}_t^{\,w}$ and $\overset{r}{\bar{\mathscr{A}}}{}_t^{\,w}$ are not necessarily unique. However $\bar{\mathscr{A}}_t^{\,+}$ is unique.

(2) Let a^g, a^r, $\overset{l}{a}{}^{w}$, $\overset{r}{a}{}^{w}$ and a^+ denote the sets of generalized inverses, reflexive generalized inverses, left weak generalized inverses, right weak generalized inverses and pseudo-inverses of matrix $\bar{\mathscr{A}}_t$ of system (II.B.3.1.a) then afore-derived results have shown sufficiently the following inclusion relationships exist:

$$a^{+} \subset a^{{}^{l}w} \subset a^{r} \subset a^{g}$$

$$a^{+} \subset a^{{}^{r}w} \subset a^{r} \subset a^{g}$$

where $\subset$ means "implies"

By use of the theorems (a)/(c) and II.B.3.1.h"/II.B.3.1.k" the numerical existence of the different concepts of a generalized inverse for the matrix $\mathcal{A}_t$ of system (II.B.3.1.o) can be easily established.

Using theorem (a) we derive:

$$B = P\,\mathcal{A}_t\,Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \qquad \text{(II.B.3.1.v')}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow E \text{ (in II.B.3.1.l) equals } 5$$

The transformation matrices of (II.B.3.1.v') appear to be orthogonal i.e.

$$P^{-1} = P^{*} \text{ and } Q^{-1} = Q^{*}.$$

Therefore it is true that:

$$P^{-1}\,B\,Q^{-1} = P^{*}\,B\,Q^{*} = \mathcal{A}_t \text{ or}$$

$$\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \qquad \text{(II.B.3.1.v")}$$

Using theorems (b) and (c) we get for the symmetric matrices

$$\mathcal{A}_t\,\mathcal{A}_t^{*} = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \text{ and } \mathcal{A}_t^{*}\,\mathcal{A}_t = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

the following results:

$$B_3 = P_3^{-1}(\mathcal{A}_t\ \mathcal{A}_t^*)P_3 = P_3^*(\mathcal{A}_t\ \mathcal{A}_t^*)P_3 =$$

$$\begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

and (II.B.3.1.w')

$$P_3\ B_3\ P_3^* = \bar{\mathcal{A}}_t\ \bar{\mathcal{A}}_t^* \text{ or}$$

$$\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}$$

moreover:

$$B_4 = P_4^{-1}(\bar{\mathcal{A}}_t^*\ \bar{\mathcal{A}}_t)P_4 = P_4^*(\bar{\mathcal{A}}_t^*\ \bar{\mathcal{A}}_t)P_4 =$$

$$\begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

and (II.B.3.1.w")

$$P_4\ B_4\ P_4^* = \bar{\mathcal{A}}_t^*\ \bar{\mathcal{A}}_t \text{ or}$$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

Subsequently we get for matrix $\bar{\mathcal{A}}_t$ of system (II.B.3.1.o):

Using theorem II.B.3.1.h" especially equation II.B.3.1.q:

$$\bar{\mathcal{A}}_t^g = Q\ B^g\ P =$$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 1/5 & a \\ b & c \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} =$$

(II.B.3.1.x')

$$\begin{bmatrix} 2/25 & 1/25 \\ 4/25 & 2/25 \end{bmatrix} + \left[\begin{bmatrix} -1/5 & -4/5 & 2/5 \\ -2/5 & 2/5 & -1/5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \vdots \begin{bmatrix} 2/5 & -2/5 & -4/5 \\ 4/5 & 1/5 & 2/5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right]$$

where a, b and c are arbitrary.

Using theorem II.B.3.1.i" especially equation (II.B.3.1.r):

$$\mathcal{A}_t^{\,r} = Q\, B^{g_1}\, B\, B^{g_2}\, P =$$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 1/5 & a_1 \\ b_1 & c_1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/5 & a_2 \\ b_2 & c_2 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \tag{II.B.3.1.x"}$$

$$\begin{bmatrix} 2/25 & 1/25 \\ 4/25 & 2/25 \end{bmatrix} \cdot \begin{bmatrix} (-4/5b_1 + 2b_1a_2 - 1/5a_2) & (-2/5b_1 - 4b_1a_2 + 2/5a_2) \\ (2/5b_1 - b_1a_2 - 2/5a_2) & (1/5b_1 + 2b_1a_2 + 4/5a_2) \end{bmatrix}$$

where a_1, b_1, c_1, a_2, b_2 and c_2 are arbitrary.

Using theorem II.B.3.1.j'" especially equation (II.B.3.1.s):

$$\overset{1}{\mathcal{A}}_t^{\,w} = (Q^{-1})^{*}\, B^{*} (P^{-1})^{*}\, P_3\, B_3^{g_1}\, B_3\, B_3^{g_2}\, P_3^{*} =$$

$$(Q^{*})^{*}\, B^{*} (P^{*})^{*}\, P_3\, B_3^{g_1}\, B_3\, B_3^{g_2}\, P_3^{*} =$$

$$Q\, B^{*}\, P\, P_3\, B_3^{g_1}\, B_3\, B_3^{g_2}\, P_3^{*} =$$

$$Q\, B^{*}\, B_3^{g_1}\, B_3\, B_3^{g_2}\, P_3^{*} = \tag{12)}$$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/25 & a_3 \\ b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/25 & a_4 \\ b_4 & c_4 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} =$$

$$\begin{bmatrix} 2/25 & 1/25 \\ 4/25 & 2/25 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} (a_4) \qquad \text{(II.B.3.1.x'")}$$

where a_3, b_3, c_3, a_4 and c_4 are arbitrary.

Using theorem II.B.3.1.j"", especially equation (II.B.3.1.t):

$$\mathcal{A}_t^{r_w} = P_4\ B_4^{g_1}\ B_4\ B_4^{g_2}\ P_4^*(Q^{-1})^*\ B^*(P^{-1})^* =$$

$$P_4\ B_4^{g_1}\ B_4\ B_4^{g_2}\ P_4^*(Q^*)^*\ B^*(P^*)^* =$$

$$P_4\ B_4^{g_1}\ B_4\ B_4^{g_2}\ P_4^*\ Q\ B^*\ P =$$

$$P_4\ B_4^{g_1}\ B_4\ B_4^{g_2}\ B^*\ P = \qquad 13)$$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 1/25 & a_5 \\ b_5 & c_5 \end{bmatrix} \cdot \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/25 & a_6 \\ b_6 & c_6 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} =$$

$$\begin{bmatrix} 2/25 & 1/25 \\ 4/25 & 2/25 \end{bmatrix} + \begin{bmatrix} -4 & -2 \\ 2 & 1 \end{bmatrix} (b_5) \qquad \text{(II.B.3.1.x"")}$$

where a_5, b_5, c_5, a_6, b_6 and c_6 are arbitrary.

Using theorem II.B.3.1.k" especially equation (II.B.3.1.u):

$$\mathcal{A}_t^{+} = (Q^{-1})^*\ B^*(P^{-1})^*\ P_3\ B_3^{g_1}\ B_3\ B_3^{g_2}\ P_3^*\ P^{-1}\ B\ Q^{-1}\ P_4\ B_4^{g_1}\ B_4\ B_4^{g_2}\ P_4^*(Q^{-1})^* .$$
$$.B^*(P^{-1})^* =$$

$$= (Q^*)^*\ B^*(P^*)^*\ P_3\ B_3^{g_1}\ B_3\ B_3^{g_2}\ P_3^*\ P^*B\ Q^*\ P_4\ B_4^{g_1}\ B_4\ B_4^{g_2}\ P_4^*(Q^*)^*\ B^*(P^*)^* =$$

$$= Q\ B^*\ P\ P_3 B_3^{g_1}\ B_3\ B_3^{g_2}\ P_3^*\ P^*\ B\ Q^*\ P_4\ B_4^{g_1}\ B_4\ B_4^{g_2}\ P_4^*\ Q^*\ B^*\ P =$$

$$= Q\, B^{*}\, B_3^{g_1}\, B_3\, B_3^{g_2}\, B\, B_4^{g_1}\, B_4\, B_4^{g_2}\, B^{*}\, P =$$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/25 & a_3 \\ b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/25 & a_5 \\ b_5 & c_5 \end{bmatrix} \cdot \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1/25 & a_6 \\ b_6 & c_6 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/25 & 1/25 \\ 4/25 & 2/25 \end{bmatrix} \qquad \text{(II.B.3.1.x""')}$$

In systems (II.B.3.1.x')/(II.B.3.1.x'"") the numerical existence of the inclusion relationships of the different concepts of a generalized inverse of the matrix $\mathcal{A}_t$ specified in system (II.B.3.1.o) has been established. Comparison of (II.B.3.1.x') and (II.B.3.1.x") shows that a reflexive generalized inverse can be constructed from a generalized inverse. Setting $a = a_2$, $b = b_1$ and $c = 5\ b_1\ a_2$ in the first system delivers the second one. Comparison of (II.B.3.1.x") and (II.B.3.1.x'") learns that a left weak generalized inverse can be constructed from a reflexive generalized inverse. Setting now $b_1 = 0$ and $a_2 = 5\ a_4$ in the first system delivers the second one. Comparison of (II.B.3.1.x'") and (II.B.3.1.x"") and (II.B.3.1.x'"") learns how the pseudo inverse or Moore-Penrose inverse can be constructed from a combination of the left weak generalized and right weak generalized inverses. Setting $a_4 = 0$ or $b_5 = 0$ in the first respectively in the second system delivers the third system of the pseudo inverse.

Transition to the pseudo inverse of the matrix $\mathcal{A}_t$ starting from one of the other concepts of a generalized inverse means reduction of the degrees of freedom. For the pseudo inverse the degrees of freedom are zero i.e. in (II.B.3.1.x.'"") the numerical existence of uniqueness of this concept of a generalized inverse has been established.

§ II.B.3.2. The Moore-Penrose Inverse technique: Numerical example and graphical presentation

In § II.B.3.1, using some topics of the linear operator theory, we got a clear insight into the relationship that exists between the concepts of the "Normal" and "Pseudo" inverses on the one side and into the relationships that exist between the Pseudo- or Moore-Penrose inverse and the other main concepts of a generalized inverse on the other side. This paragraph will be divided into two subparagraphs of which the first one will be devoted to the existence of the 'General' respectively 'Unique' solutions of the normalized inverted optimization system $\bar{\mathcal{A}}_t \cdot \bar{p}_t = g_t$ in case of consistency and of the "Best Approximate" solution of this system in case of inconsistency.
In subparagraph II.B.3.2.2 we can turn our attention again to system (II.B.3.1.a) in its numerically specified form as denoted by system (II.B.3.1.o) and apply the results so far derived. The latter subparagraph will be closed with the graphical presentation of the 'General' and 'Unique' solution of system (II.B.3.1.o).

§ II.B.3.2.1. Existence of the 'General', 'Unique' and 'BAS' solutions of the 'Normalized' inverted optimization problem

Early in this section we suggested that a necessary and sufficient condition for the matrix equation of the normalized inverted optimization problem (II.B.3.1.a) i.e.

$$\bar{\mathcal{A}}_t \cdot \bar{p}_t = g_t \qquad \text{(II.B.3.2.1.a)}$$

to have a solution (i.e. corresponding to a consistent system) is:

$$\bar{\mathcal{A}}_t \, \bar{\mathcal{A}}_t^{+} \, g_t = g_t \qquad \text{(II.B.3.2.1.b)}$$

in which case the 'General' solution is:

$$\bar{p}_t = \bar{p}_{t,o} = \bar{\mathcal{A}}_t^{+} \, g_t + (I - \bar{\mathcal{A}}_t^{+} \, \bar{\mathcal{A}}_t)\underline{r} \qquad \text{(II.B.3.2.1.c)}$$

with arbitrary $\underline{r} \in R^n$.
This can be proved as follows.

If $\bar{\underline{p}}_t$ satisfies $\mathcal{A}_t \bar{\underline{p}}_t = \underline{g}_t$, then using the results of the foregoing paragraph:

$$\underline{g}_t = \mathcal{A}_t \bar{\underline{p}}_t = \mathcal{A}_t \mathcal{A}_t^+ \mathcal{A}_t \bar{\underline{p}}_t = \mathcal{A}_t \mathcal{A}_t^+ \underline{g}_t$$

and conversely, if $\mathcal{A}_t \mathcal{A}_t^+ \underline{g}_t = \underline{g}_t$ then $\mathcal{A}_t^+ \underline{g}_t = \bar{\underline{p}}_{t,u}$ is a particular solution of $\mathcal{A}_t \bar{\underline{p}}_t = \underline{g}_t$.
Besides $\forall_{\underline{r}}\ \bar{\underline{p}}_{t,N} = (I - \mathcal{A}_t^+ \mathcal{A}_t)\underline{r}$ satisfies the relation

$$\mathcal{A}_t \bar{\underline{p}}_{t,N} = \underline{0}$$

and conversely if $\mathcal{A}_t \bar{\underline{p}}_{t,N} = \underline{0}$ then $\bar{\underline{p}}_{t,N} = \bar{\underline{p}}_{t,N} - \mathcal{A}_t^+ \mathcal{A}_t \bar{\underline{p}}_{t,N}$.
From this it follows that

$$\bar{\underline{p}}_t = \bar{\underline{p}}_{t,o} = \bar{\underline{p}}_{t,u} + \bar{\underline{p}}_{t,N} = \mathcal{A}_t^+ \underline{g}_t + (I - \mathcal{A}_t^+ \mathcal{A}_t)\underline{r}$$

is the 'General' solution of $\mathcal{A}_t \bar{\underline{p}}_t = \underline{g}_t$

Moreover we noted in the case of consistency of system (II.B.3.2.1.a) that $\bar{\underline{p}}_{t,u} = \mathcal{A}_t^+ \underline{g}_t$ is a particular solution, to be considered later on as the 'best' solution.

In case of inconsistency of system (II.B.3.2.1.a)

$$\bar{\underline{p}}_{t,\hat{o}} = \mathcal{A}_t^+ \underline{g}_t$$

was the "Best Approximate" solution. Both theorems can be easily proved.

Let $\bar{\mathcal{A}}_t$ be a (m×n)-matrix of system (II.B.3.2.1.a) and $\|\bar{\mathcal{A}}_t\|_2$ denote the non-negative square root of the sum of squares of the moduli of the elements of $\bar{\mathcal{A}}_t (\triangleq \|\bar{\mathcal{A}}_t\|^2)$.

It is true that $\|\bar{\mathcal{A}}_t\|^2 = \text{tr.}\ \bar{\mathcal{A}}_t^* \ \bar{\mathcal{A}}_t$ and $\|\bar{\mathcal{A}}_t\|_2 > 0$ unless $\bar{\mathcal{A}}_t = 0$, then $\|\bar{\mathcal{A}}_t\|_2 = 0$.

We can say by definition that the vector $\bar{p}_{t,\hat{o}}$ is a 'Best approximate' solution of system (II.B.3.2.1.a) if for all $\bar{p}_t \neq \bar{p}_{t,\hat{o}}$ either

$$\|\bar{\mathcal{A}}_t \ \bar{p}_t - g_t\|_2 > \|\bar{\mathcal{A}}_t \ \bar{p}_{t,\hat{o}} - g_t\|_2 \neq \underline{0}$$

or

$$\|\bar{\mathcal{A}}_t \ \bar{p}_t - g_t\|_2 = \|\bar{\mathcal{A}}_t \ \bar{p}_{t,\hat{o}} - g_t\|_2 \neq \underline{0}$$

and

$$\|\bar{p}_t\|_2 > \|\bar{p}_{t,\hat{o}}\|_2$$

We can also say by defition that $\bar{p}_{t,u}$ is a 'Best' solution of system (II.B.3.2.1.a) if for all $\bar{p}_t = \bar{p}_{t,o}$

$$\|\bar{\mathcal{A}}_t \ \bar{p}_t - g_t\|_2 = \|\bar{\mathcal{A}}_t \ \bar{p}_{t,u} - g_t\|_2 = \underline{0}$$

and

$$\|\bar{p}_t\|_2 \geq \|\bar{p}_{t,u}\|_2$$

Why it is the 'best' solution with regard to the D.S.I.D.-model will be clarified in next paragraph.

<u>Now we derive</u>:

$$\|\bar{\mathcal{A}}_t \ \bar{p}_t - g_t\|^2 = \|\bar{\mathcal{A}}_t(\bar{p}_t - \bar{\mathcal{A}}_t^+ \ g_t) + (I - \bar{\mathcal{A}}_t \ \bar{\mathcal{A}}_t^+)(-g_t)\|^2$$

$$= \|\bar{\mathcal{A}}_t(\bar{p}_t - \bar{\mathcal{A}}_t^+ \ g_t)\|^2 + \|(I - \bar{\mathcal{A}}_t \ \bar{\mathcal{A}}_t^+)(-g_t)\|^2$$

$$= |\mathcal{A}_t \bar{p}_t - \mathcal{A}_t \mathcal{A}_t^+ g_t)|^2 + |\mathcal{A}_t \mathcal{A}_t^+ g_t - g_t|^2$$

$$\geq |\mathcal{A}_t \mathcal{A}_t^+ g_t - g_t|^2$$

and

$$|\mathcal{A}_t \bar{p}_t - g_t)|^2 = |\mathcal{A}_t \mathcal{A}_t^+ g_t - g_t|^2$$

only if

$$|\mathcal{A}_t \bar{p}_t - \mathcal{A}_t \mathcal{A}_t^+ g_t|^2 = 0 \rightarrow$$

$$|\mathcal{A}_t \bar{p}_t - \mathcal{A}_t \mathcal{A}_t^+ g_t|_2 = 0$$

or

$$\mathcal{A}_t \bar{p}_t = \mathcal{A}_t \mathcal{A}_t^+ g_t$$

From the latter equality (see also (II.B.3.2.1.b)) it can be proved that $\mathcal{A}_t^+ g_t$ is the 'Best Approximate' solution or the 'Best' solution defined as before.

For $\bar{p}_t = \mathcal{A}_t^+ g_t + (I - \mathcal{A}_t^+ \mathcal{A}_t)\bar{p}_t$ we derive

$$|\mathcal{A}_t^+ g_t + (I - \mathcal{A}_t^+ \mathcal{A}_t)\bar{p}_t|^2 = |\mathcal{A}_t^+ g_t|^2 + |(I - \mathcal{A}_t^+ \mathcal{A}_t)\bar{p}_t|^2$$

For the equality $\mathcal{A}_t \bar{p}_t = \mathcal{A}_t \mathcal{A}_t^+ g_t$ it is true that

$$|\mathcal{A}_t^+ g_t|^2 + |(I - \mathcal{A}_t^+ \mathcal{A}_t)\bar{p}_t|^2 =$$

$$|\mathcal{A}_t^+ g_t|^2 + |\bar{p}_t - \mathcal{A}_t^+ g_t|^2 = |\bar{p}_t|^2$$

so:

$$\min|\bar{p}_t|^2 = |\mathcal{A}_t^+ g_t|^2 \text{ i.e.}$$

$$|p_t - \mathcal{A}_t^+ g_t|^2 = 0 \Rightarrow p_t - \mathcal{A}_t^+ g_t = \underline{0} \Rightarrow$$

$$\bar{p}_t = \mathcal{A}_t^+ g_t \qquad \text{q.e.d.}$$

§ II.B.3.2.2. Numerical Example and Graphical Presentation

System (II.B.3.2.1.a) numerically specified as we did in system (II.B.3.1.o), must be considered as a consistent system. So the "General" solution is given by:

$$\bar{p}_{t,o} = \mathcal{A}_t^+ \, g_t + (I - \mathcal{A}_t^+ \, \mathcal{A}_t)\underline{r} \qquad \text{(II.B.3.2.2.a)}$$

whereas the 'unique' solution is found by

$$\bar{p}_{t,u} = \mathcal{A}_t^+ \, g_t \qquad \text{(II.B.3.2.2.b)}$$

In the foregoing paragraph we derived already the Moore-Penrose Inverse $\mathcal{A}_t^+$ of the corresponding matrix $\mathcal{A}_t$ of system (II.B.3.1.o) in an algebraical way.
Many techniques respectively algorithms for obtaining this Moore-Penrose inverse on a computer have appeared in the literature. These algorithms are often based on linear operator theory without taking into account problems in the field of pertubation theory. One such algorithm is re-stated in an earlier paper where its application is performed on the same matrix $\mathcal{A}_t$ of this numerical example from which it will be clear we can get $\mathcal{A}_t^+$ in only five steps.[14)]
Comparison with the algebraic method of the foregoing paragraph learns that such an iterative procedure can curtail the computatons considerably. However vexing problems in the field of the pertubation theory do not always allow us to use the latter recursive method for the D.S.I.D.-application model. Anyway for the computation the $\mathcal{A}_t^+$-matrix, in order to get $\bar{p}_{t,o}$ and $\bar{p}_{t,u}$ of the theoretical D.S.I.D.-model, where we can abstract from the possibility of pertubation of the matrix $\mathcal{A}_t$ and of the vector g_t, can it be used as we did before in the example demonstrated in an earlier research memorandum.[15)]

Substitution of $\mathcal{A}_t^+$ of (II.B.3.1.x""') into (II.B.3.2.2.a) results into:

$$\begin{bmatrix} \bar{p}_{1,t,o} \\ \bar{p}_{2,t,o} \end{bmatrix} = \begin{bmatrix} 2/25 & 1/25 \\ 4/25 & 2/25 \end{bmatrix} . \begin{bmatrix} 10 \\ 5 \end{bmatrix} + \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/25 & 1/25 \\ 4/25 & 2/25 \end{bmatrix} . \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \right] . \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} . \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 + 4/5\, r_1 - 2/5\, r_2 \\ 2 - 2/5\, r_1 + 1/5\, r_2 \end{bmatrix} \qquad \text{(II.B.3.2.2.c)}$$

From (II.B.3.2.2.b) we derive:

$$\begin{bmatrix} \bar{p}_{1,t,u} \\ \bar{p}_{2,t,u} \end{bmatrix} = \begin{bmatrix} 2/25 & 1/25 \\ 4/25 & 2/25 \end{bmatrix} . \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{(II.B.3.2.2.d)}$$

The latter result (II.B.3.2.2.d) can be derived from (II.B.3.2.2.c) i.e.

$$\min \left\| \begin{bmatrix} \bar{p}_{1,t,o} \\ \bar{p}_{2,t,o} \end{bmatrix} \right\|_2 = \min \sqrt{\bar{p}^2_{1,t,o} + \bar{p}^2_{2,t,o}} =$$

$$\min_{r_1, r_2} [(1 + 4/5\, r_1 - 2/5\, r_2)^2 + (2 - 2/5\, r_1 + 1/5\, r_2)^2]^{\frac{1}{2}} = 5$$

for $r_1 = \frac{1}{2} r_2$

The latter solution for the original arbitrary r's denotes a range of solutions, each element of which satisfies $r_1 = \frac{1}{2} r_2$ and can be used to obtain now the unique solution $\bar{\underline{p}}_{t,u}$.

Replacing the $\underline{r}$-vector in system (II.B.3.2.2.c) taking into account the condition $r_1 = \frac{1}{2} r_2$ we get:

$$\bar{\underline{p}}_{t,u} = \begin{bmatrix} \bar{p}_{1,t,u} \\ \bar{p}_{2,t,u} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

From this it is clear again the $\bar{p}_{t,u}$-vector is a least-least-squares solution vector.

Figure II.B.3.2.2.1 is the graphical presentation of (II.B.3.2.2.c) and (II.B.3.2.2.d).

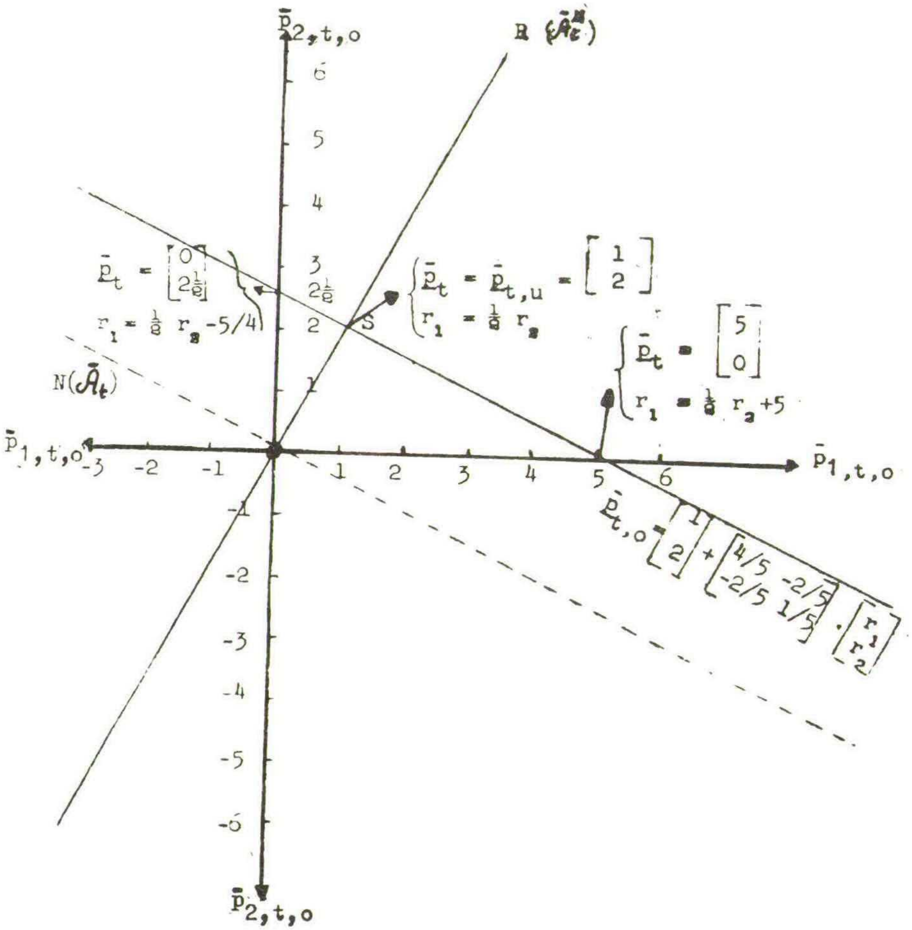

Figure II.B.3.2.2.1

From figure II.B.3.2.2.1 we can conclude that the straight line denoting all feasible solutions (= 'General' solution) of system (II.B.3.2.2.c) runs parallel to the shattered straight line indicating the null-space of the linear transformation $\mathcal{A}_t$ of system (II.B.3.1.o). These two lines are perpendicular to the straight line denoting $R(\mathcal{A}_t^*)$. The least euclidean distance between $\bar{p}_{t,o}$ of system (II.B.3.2.2.c) and the origine of figure II.B.3.2.2.1 is given by point S. In this point the unique vector $\bar{p}_{t,u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and is thus defined as being the intersection point S of the two straight lines $R(\mathcal{A}_t^*)$ and $\bar{p}_{t,o}$.

§ II.B.4. Final framework and evaluation of the D.S.I.D.-model

Until now we elaborated the basic framework of the theoretical D.S.I.D.-model. In this paragraph we shall set up the final framework of the model and give an evaluation of it.
The latter consists of a justification of the main elements of our way of doing f.i. with regard to the synthesis of the Lagrange Multiplier and the Moore-Penrose Inverse techniques.
Besides shall discuss the usefulness of the theoretical results for setting up the D.S.I.D.-application model.

§ II.B.4.1. Setting up the final framework

The basic framework of the theoretical D.S.I.D.-model is represented by the matrix equation (II.B.2.2.c) of the normalized inverted optimization problem. Using the results developed in § II.B.2.3 and § II.B.3, the final framework can be set up as follows.

From systems (II.B.2.2.c) and (II.B.2.3.e) we derive:

$$\forall_t^{t=1,\dots,T}$$

$$\begin{bmatrix} \\ \underline{\bar{p}}_{t,u} \\ \\ \end{bmatrix} = \begin{bmatrix} \underline{\omega}_{\underline{y}_{t,u}}(\underline{y}(0)_t, \underline{z}(0)_t) \\ \underline{\omega}_{\underline{z}_{t,u}}(\underline{y}(0)_t, \underline{z}(0)_t) \\ \underline{\bar{\lambda}}_{t,u} \end{bmatrix} = \left[\begin{array}{c|c} I & \begin{array}{c} -\bar{A}_t' \\ -\bar{C}_t' \end{array} \\ \hline 0 & -\bar{B}_t' \end{array}\right]^{+} + \begin{bmatrix} \underline{a}_{1,t}' \\ \underline{c}_{1,t}' \\ \underline{b}_{1,t}' \end{bmatrix} \qquad \text{(II.B.4.1.a)}$$

with the $\underline{\bar{p}}_{t,u}$ vector of which the first (J+K) elements can be expressed in relative terms of each other (if the denominator $\neq 0$). By this $\frac{(J+K)!}{2!(J+K-2)!}$ values connecting with the different ratios (of the marginal preferences) of the target- and instrumental variables with respect to each other are found for every year t. Multiplication of those 'relative preference ratios' by the corresponding reciprocal ratios of the realized values of the target and/or instrumental variables results in the 'relevant' values of the 'relative preference elasticity ratios' of the target- and instrumental variables for every year t.

Any ratio value with regard to a relative preference elasticity for a certain year t can be calculated for all T years of the observation horizon and so the evolution over time can be considered by observing these calculated realizations and their corresponding curves.

Comparison of the latter ones with the curves corresponding to the retrospective values, to be generated by getting the most adequate polynomial to the different ratios during the observation horizon allows to detect the nature of the evolutions over time of the relative preference elasticities, and to specify the dynamic properties of the preference structure. How far the most-adequate polynomial fitted biases the original D.S.I.D-model results against the H_0-hypothesis (stability of the corresponding relative preference elasticities) will be exposed elsewhere.[5] There we shall demonstrate that the goodness of fit i.c. the choice of the k^{th} order curve linear model:

$$\frac{\bar{p}_{i,u}}{\bar{p}_{j,u}} = f(t) = \vartheta_0 + \vartheta_1 t + \vartheta_2 t^2 \dots\dots\dots \vartheta_k t^k$$

is based on variance analysis using F-test.
Further insight into the dynamic property of a given preference elasticity can be got by means of an analysis of the deviations between the calculated original D.S.I.D-model results and the retrospective results to be generated by the corresponding polynomial.

§ II.B.4.2. Justification

Fundamental justification of the theoretical D.S.I.D.-model approach must concern our way of solving problems in view of the ultimate purpose of our investigation viz. our wish to detect the evolution through time of the preference structure characterizing an economy. This preference structure is defined as the set of 'realized' relative preferences - respectively of 'realized' relative preference elasticity ratios for a certain year t. The underlying problems belong to the fields of different scientific disciplines e.g. those of economic-political theory, of mathematics and of statistics. Because we believe many aspects are clarified in the foregoing paragraphs, we shall constrain ourselves to give a justification of the second main element built in in our D.S.I.D.-model, i.e. that regarding the second part of the synthesis of the Lagrange multiplier and the Moore-Penrose inverse techniques.

As we saw, the D.S.I.D.-model allows for more than one solution in case of consistency. So we are dealing with more than one set of 'realized' relative preference- and preference elasticity ratios being all feasible for a certain year t.
The ultimate purpose of our investigation can be re-formulated now as follows:

A. We want to derive the whole set of feasible preference structures for a certain year t during an observation horizon of T years with regard to a certain economy.

B. We are not seeking for the 'actual' preference structure in the first place but for the tendency of the set of feasible preference structures to change or not.

From A and B it will be clear that the ultimate use of the solution vector $\bar{\underline{p}}_{t,u}\ \forall_t^{t=1,\ldots,T}$ of system (II.B.4.1.a) and its underlying selection procedure must be proved to be 'most useful' compared with other selection procedures normally used in the literature concerning such choice problems as picking one solution vector from the feasible set of solution vectors of a consistent system as we have in (II.B.2.2.c).[16)] 'Most useful' refers to the claim that ratios of the corresponding solution values are the 'best relative preference indicators' and are to be very sensitive to variations in the known elements referring to a certain year t which biases them ultimately against the H_o-hypothesis: 'stability of relative preferences' in the theoretical D.S.I.D.-model. They are 'best indicators' in the sense that their own variations are exactly representative for variations of the range of feasible sets of ratios as a whole. Besides we must demonstrate why the Moore-Penrose inverse idea takes care in particular for satisfying aforementioned conditions.

In the literature the mostly used selection procedures in choice problems, such as we are handling with are based on any measure of 'how close the solution values deviate from zero' i.e. any measure from the family of so-called Lp-metrics (or norms).

This family consists of a set of real valued functions, $|\ |$, on R^n satisfying certain conditions. In fact this family of functions brings to mind the well known characteristics of a CES prodution function i.e. in terms of our $\bar{\underline{p}}_{t,o}$-vector elements the set:

$$Lp(\bar{\underline{p}}_{t,o}) = |\bar{\underline{p}}_{t,o}|_p = \left[\sum_{i=1}^{J+K+N-1} |\bar{p}_{i,t,o}|^p\right]^{1/p} \quad p \geq 1 \qquad \text{(II.B.4.2.a)}$$

where n = J+K+N-1.

The most popular Lp-norms on which the selection procedures are based are the choices p = 1,2 and ∞ in (II.B.4.2.a) i.e.

(a) The procedure consisting of minimizing the sum of feasible values (in absolute terms) of the elements of the vector $\bar{\underline{p}}_{t,o}$ of (II.B.3.2.2.a). It boils down to minimizing $Lp(\bar{\underline{p}}_{t,o})$ of (II.B.4.2.a) for p = 1. i.e.

$$\min|\bar{\underline{p}}_{t,o}|_1 = \min \sum_{i=1}^{J+K+N-1} |\bar{p}_{i,t,o}|$$

(b) The procedure consisting of the choice of that vector of which the highest valued element is the smallest one among the highest valued elements of the other feasible valued solution vectors of (II.B.3.2.2.a), where all values are taken in absolute terms. It boils down to minimization of $Lp(\bar{p}_{t,o})$ of (II.B.4.2.a) for $p = \infty$ i.e.:

$$\min\|\bar{p}_{t,o}\|_{\infty} = \text{min-max}\{/\bar{p}_{i,t,o}/: \quad i = 1,\ldots,J+K+N-1\}$$

also called the Tchebycheff min-max method.

(c) The L.L.S.S. or minimum Euclidean norm solution procedure. It boils down to minimization of $Lp(\bar{p}_{t,o})$ of (II.B.4.2.a) for $p = 2$ i.e.

$$\min\|\bar{p}_{t,o}\|_{2} = \min\left[\sum_{i=1}^{J+K+N-1}/p_{i,t,o}/^{2}\right]^{\frac{1}{2}}.$$

These three procedures of making a choice for $\bar{p}_t$ of system (II.B.3.2.1.a) from the set of solution vectors $\bar{p}_{t,o}$ of (II.B.3.2.2.a) in the case of consistency can be exposed by considering the figures II.B.4.2.1, II.B.4.2.2 and II.B.4.2.3.

Let we have for system (II.B.3.2.1.a) the following numerical specifications:

<u>Situation I</u>: $t = 1$

$$\mathcal{A}_1 \cdot \bar{p}_1 = q_1 \Rightarrow \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \bar{p}_{1,1} \\ \bar{p}_{2,1} \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad \text{(II.B.4.2.4)}$$

<u>Situation II</u>: $t = 2$

$$\mathcal{A}_2 \cdot \bar{p}_2 = q_2 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \bar{p}_{1,2} \\ \bar{p}_{2,2} \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad \text{(II.B.4.2.5)}$$

using the 'General-Solution' equation system (II.B.3.2.2.a) we get for <u>situation I</u>:

$$\begin{matrix} \bar{p}_{1,1_o} \\ \bar{p}_{2,1_o} \end{matrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \qquad \text{(II.B.4.2.6)}$$

and for <u>situation II</u>:

$$\begin{matrix} \bar{p}_{1,2_o} \\ \bar{p}_{2,2_o} \end{matrix} = \begin{bmatrix} 2\frac{1}{2} \\ 2\frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \qquad \text{(II.B.4.2.7)}$$

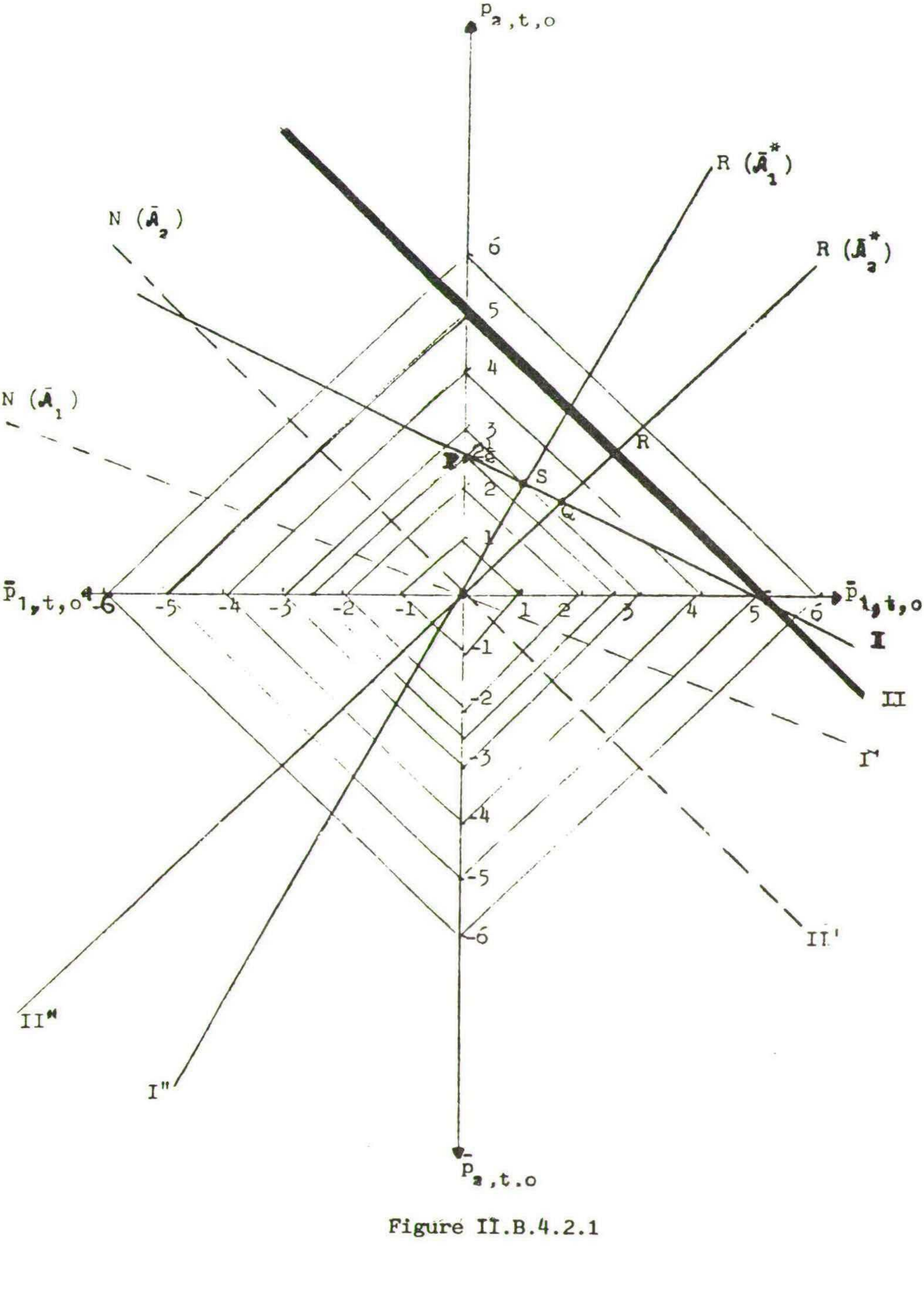

Figure II.B.4.2.1

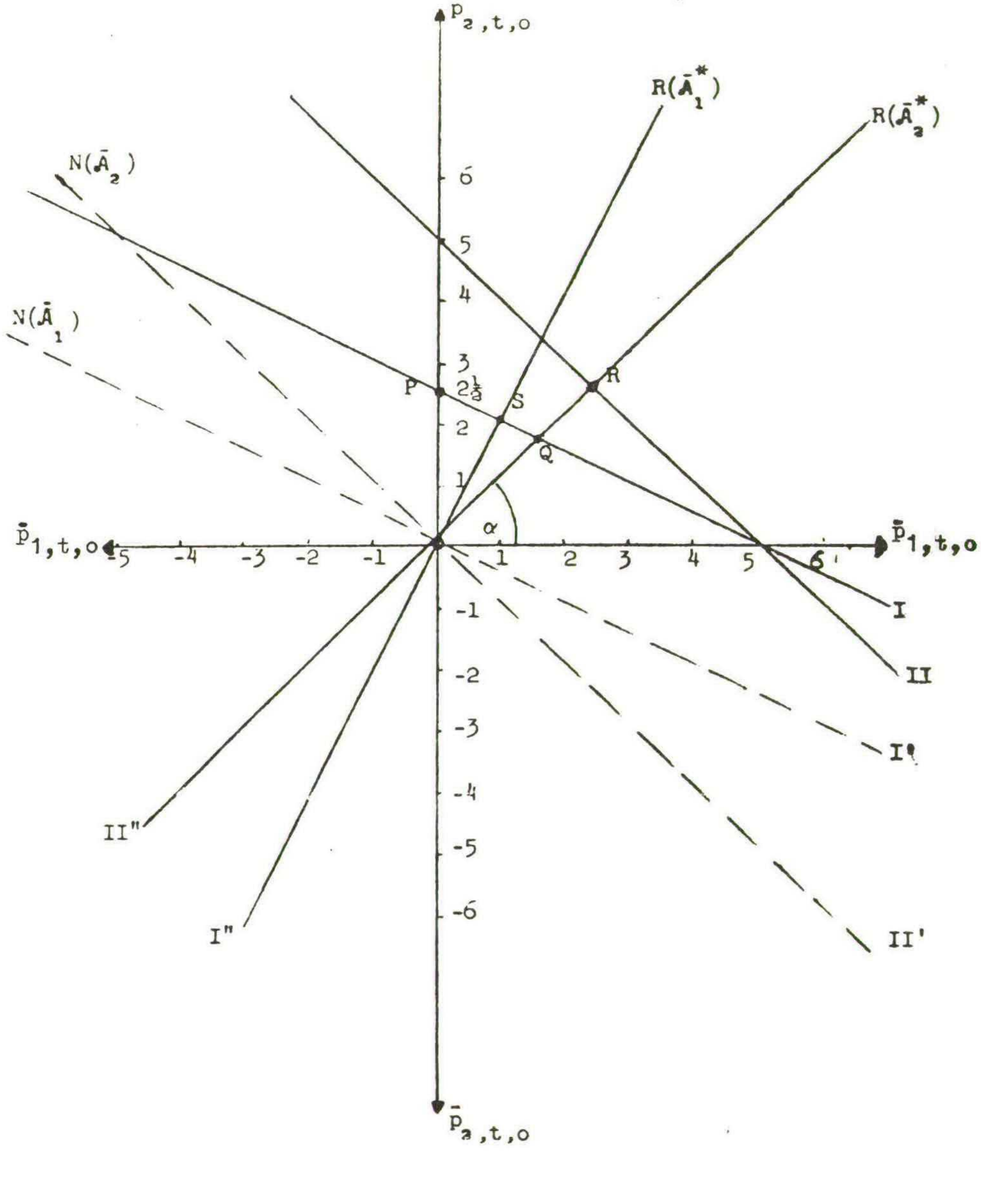

Figure II.B.4.2.2

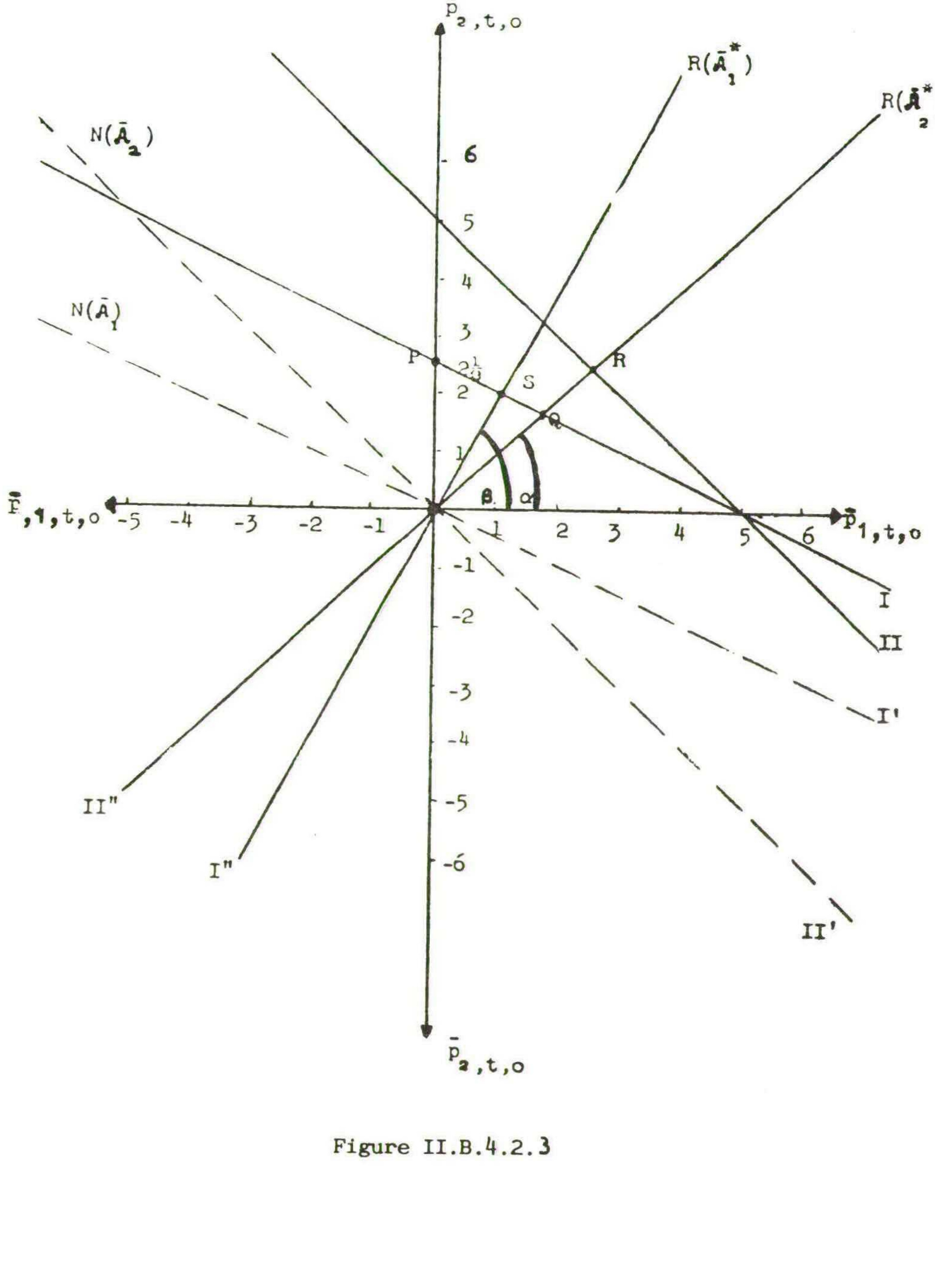

Figure II.B.4.2.3

Now we are able to consider the consequences of handling the diverse selection procedures (a), (b) and (c) in order to find an 'unique' vector $\bar{p}_{t,u}$ from the set of possible solution vectors $\bar{p}_{t,o}$ of the systems (II.B.4.2.6) and (II.B.4.2.7) i.e.

sub (a): Procedure (a) results into:

$$\begin{bmatrix} \bar{p}_{1,1,u} \\ \bar{p}_{2,1,u} \end{bmatrix} = \begin{bmatrix} 0 \\ 2\frac{1}{2} \end{bmatrix} \qquad \text{for situation I}$$

(see: point P of Figure II.B.4.2.1)

and into a set of solutions $\bar{p}_{2,u}$ for situation II coinciding with the set of solutions $\bar{p}_{2,o}$ of system (II.B.4.2.7)
(see: Straight line II in figure II.B.4.2.1)

sub (b): Procedure (b) results into:

$$\begin{bmatrix} \bar{p}_{1,1,u} \\ \bar{p}_{2,1,u} \end{bmatrix} = \begin{bmatrix} 1\frac{2}{3} \\ 1\frac{2}{3} \end{bmatrix} \qquad \text{for situation I}$$

(see: point Q of Figure II.B.4.2.2)

and into:

$$\begin{bmatrix} \bar{p}_{1,2,u} \\ \bar{p}_{2,2,u} \end{bmatrix} = \begin{bmatrix} 2\frac{1}{2} \\ 2\frac{1}{2} \end{bmatrix} \qquad \text{for situation II}$$

(see: point R of Figure II.B.4.2.2)

sub (c): Procedure (c) results into:

$$\begin{bmatrix} \bar{p}_{1,1,u} \\ \bar{p}_{2,1,u} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ for situation I

(see: point S of Figure II.B.4.2.3)

and into:

$$\begin{bmatrix} \bar{p}_{1,2,u} \\ \bar{p}_{2,2,u} \end{bmatrix} = \begin{bmatrix} 2\frac{1}{2} \\ 2\frac{1}{2} \end{bmatrix}$$ for situation II

(see: point R of Figure II.B.4.2.3)

With regard to the consequences of handling with one of the three selection procedures (a), (b) and (c) as exposed in sub (a), sub (b) and sub (c) respectively in the figures II.B.4.2.1/II.B.4.2.3 we can conclude that if the general system (II.B.3.2.1.a) is consistent and more than one solution exists, as we have in situations I and II, the latter ones being real possibilities with regard to our D.S.I.D.-model system (after premultiplying its left and right members with the transposed $\bar{\mathcal{A}}_t$-matrix) than:

1.a.: Procedure (a) will not be appropriate taking into account our ultimate purpose to detect the evolution over time of any ratio of the first J + K elements of a unique vector solution. This is because this procedure allows for the possibilities of getting an ultimate solution vector not being a unique one but a set of feasible solution vectors as we had already for $\bar{p}_{t,o}$ of system (II.B.3.2.2.a) (see: straight line II of figure II.B.4.2.1). Besides if the ultimate vector to be found by means of this procedure is a unique one it will always indicate a corner point the consequence of which being that the different ratios between the elements of this vector containing at least one zero valued element are zero or undetermined.

So we may say that procedure (a) is not a suitable one because it only generates solution vectors which are described geometrically as being points on the straight lines of the 'Cob-Web' of figure II.B.4.2.1. (see: Point P of Figure II.B.4.2.1 denoting the unique solution vector $\bar{p}_{t,u} = \begin{bmatrix} 0 \\ 2\frac{1}{2} \end{bmatrix}$; the cotangent of the angle between the straight lines 0 - P and 0 - $\bar{p}_{1,t,o}$ gives the ratio between $\bar{p}_{1,1,u}$ and $\bar{p}_{2,1,u}$)

1.b.: Procedure (b) will always give solution vectors $\bar{p}_{t,u}$ for which the ratios between its real-valued elements equal to one. So it can not be an appropriate one because it excludes a priori changes of the diverse ratios.

The cotangent α in figure II.B.4.2.2 remains one from period $t = 1$ to period $t = 2$. This circumstance means that in spite of the fact the situation has been changed taking into account the changed set of feasible solutions of the D.S.I.D.-model from one period to another the Tchebycheff Min-Max-method will not indicate it.

1.c.: Procedure (c) can be qualified as being 'powerful' with regard to the purpose of seeking for the time properties of the different ratios of the set of feasible solutions (= the general solution) to the D.S.I.D.-model.

Points S and R in Figure II.B.4.2.3 denote the $\bar{p}_{t,u}$ vectors in period 1 respectively in period 2.

The cotangents of β and α give the ratios of the elements of these two unique vectors. They indicate a change from period 1 to period 2 of the general solution being the set of feasible solutions of the general system (II.B.3.2.1.a).

In how far we are allowed to use the latter results in analogue cases for the D.S.I.D.-model in order to conclude the preference structure of the economy has been changed still remains to be clarified.

At any rate we can say that the demonstrated use of the solutions to be found to the D.S.I.D.-model by means of procedure (c), i.e. calculating the ratios of the valued elements of the $\bar{p}_{t,u}$ vector, is very sensitive to variations in the known elements referring to a certain year t and more preferable than the results of procedure (a) or (b).

Because for the theoretical D.S.I.D.-model system we are dealing with the case of consistency, procedure (c) does not give actually a least-norm-least-squares solution but a least-norm solution i.e. the least-Euclidean norm solution.

The latter means that this solution for the theoretical D.S.I.D.-model, i.e. $\bar{p}_{t,u}$, does not ask for the use of the Moore-Penrose inverse $\mathcal{A}_t^{+}$ satisfying simultaneously the four conditions II.B.3.1.h/II.B.3.1.k.

A generalized inverse of the matrix $\mathcal{A}_t$ satisfying only conditions II.B.3.1.h and II.B.3.1.j can give the solution after multiplication with the g_t vector. The same is true for a left weak generalized inverse $\mathcal{A}_t^{1w}$ satisfying these two conditions and moreover condition II.B.3.1.i. This can be easily verified if we consider again the above example of situation I. Multiplying the vector $g_t = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$ by expression (II.B.3.1.x""') as well as by expression (II.B.3.1.x"') will give the same results i.e. $\bar{p}_{t,u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The early use of the Moore-Penrose Inverse in the theoretical D.S.I.D.-model has two reasons:

(1) For the sake of completeness we had to use it viz. if the D.S.I.D.-model system would become inconsistent, f.i. as a consequence of rounding off errors in performing the computation of the original $\mathcal{A}_t$-matrix, only the Moore-Penrose inverse would result in getting the minimum-Euclidean norm least-squares solution performing procedure (c). This so-called Best-approximate solution has the same advantages as we described herebefore with regard to changes from period to period.

(2) Another possibility for rounding off errors gives rise to the use the Moore-Penrose-Inverse in the D.S.I.D.-application model in the forthcoming paper. Because of the possibility that the D.S.I.D.-model equations and its matrix $\mathcal{A}_t$ are ill-conditioned i.e. that the solutions are very sensitive to small changes in the data, the same can happen with regard to rounding off errors as a consequence of the numerical computation of the solution itselves.

Methods for computing in this case minimum-Euclidean norm least squares solutions which take account of this difficulty are based on the Moore-Penrose Inverse technique; however they are different from the method that we used earlier and where such rounding off errors are bypassed.

§ II.B.4.3. Use of the theoretical D.S.I.D.-model results for setting up the D.S.I.D.-application model

Until now it has been clear to what extent the theoretical D.S.I.D.-model allows for investigations on stability of preference structures dominating an economy. It merely depends on assumptions underlying the synthesis of the Lagrange multiplier and 'Pseudo' Inverse techniques.
Use of the Lagrange multiplier technique was justified by postulating the underlying assumptions of quantitative economic policy and the existence of a global maximum of an objective function of the policy decision unit are met perfectly.
Use of the D.S.I.D.-model solution obtained by use of the Moore-Penrose Inverse technique was merely justified by the point of view of a good test on stability in the sense we already indicated. This justification may appear less sophisticated if we accept on theoretic-economic-political grounds the a priori idea that relative preference ratios do not change rapidly from year to year. However this latter justification should be considered only as an implementation if we realize again we are not interested in the preference structure itselves in the first place but in its tendency to change or not.

If we shall develop the D.S.I.D.-application model in a forthcoming paper one main difference with regard to the theoretical D.S.I.D.-model results will appear:

Whereas the theoretical D.S.I.D.-model generates data to be transformed into 'relative preference data' where after these 'relative preference data' could be transformed into 'relative preference elasticity data', the D.S.I.D.-application model generates immediately the latter data by weighting the first J + K elements of the $\bar{p}_{t,u}$-vector.

II.C. Notes

1) see the bibliography, numbers 20, 21 and 22.

2) see the bibliography, number 22.

3) see the bibliography, number 21.

4) see the bibliography, numbers 20, 21, 22 and 31.

5) see two other Research Memoranda: An exercise in welfare economics IV and V; forthcoming.

6) see the bibliography, especially that I have quoted earlier in the references of the Research Memoranda, numbers 21 and 22.

7) see the bibliography, in particular the numbers 5, 11, 12, 13, 24, 26, 29 and 30.

8) Different indices g_1 and g_2 denote the possibility that the numerical values of the elements of the corresponding matrices α, β and γ are different.

9) Matrix $\bar{\mathscr{A}}_t$ is not the same matrix $\bar{\mathscr{A}}_t$ of theorem (c).

10) If all eigenvalues are non-null implies B_3 is a non-singular diagonal matrix and $B_3^g = B_3^{-1}$.

11) If all eigenvalues are different from zero B_4 is a non-singular matrix and $B_4^g = B_4^{-1}$.

12) In this special case it is true that $P_3 = P^* = P^{-1} \Rightarrow PP_3 = PP^* = PP^{-1} = I$. Besides $Q^{-1} = Q^*$.

13) In this special case it is true that $P_4 = Q$; $Q^{-1} = Q^*$ and $P^{-1} = P^*$, so $P_4^*Q = Q^*Q = Q^{-1}Q = I$.

14) see the bibliography, number 20.

15) see the bibliography, number 21.

16) see the bibliography, in particular number 32.

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